1 Let F be a field and let $R \subseteq F$ be a subring such that $1 \in R$. Prove that R is an integral domain.

2 Let F be a field and let $R = F[x,y]$ be the ring of polynomials in two variables with coefficients in F.

(a) Prove that

$$\begin{aligned} \mathrm{ev}_{(0,0)} \colon F[x,y] &\to F \\ p(x,y) &\mapsto p(0,0) \end{aligned}$$

is a surjective ring homomorphism.

(b) Prove that $\ker \mathrm{ev}_{(0,0)}$ is equal to the ideal

$$(x,y) = \{xr(x,y) + ys(x,y) \mid r,s \in F[x,y]\}$$

(c) Use the first isomorphism theorem to prove that $(x,y) \subseteq F[x,y]$ is a maximal ideal.

(d) Find an ideal $I \subseteq F[x,y]$ such that I is prime but not maximal. [HINT: Find a surjective homomorphism $F[x,y] \to F[x]$.]

(e) Find an ideal $J \subseteq F[x,y]$ such that J is not prime.

3 Let F be a field and n a positive integer. Show that if $\omega \in F$ satisfies $\omega^n = 1$, then either $\omega = 1$ or $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$.

4 Use Cardano's method to find all complex solutions to

$$x^3 + 6x^2 + 18x + 18 = 0.$$