1 Let F be a field and $p(x), q(x) \in F[x]$. Show that the following are equivalent:

(a) $p(x)$ divides $q(x)$.

(b) $q(x) \in (p(x))$.

(c) $(q(x)) \subseteq (p(x))$.

2 Let R be a commutative ring with 1 and let a and b be nonzero elements of R. A **least common multiple** of a and b is an element $\ell \in R$ such that

- a divides ℓ and b divides ℓ; and
- if a divides ℓ' and b divides ℓ', then ℓ divides ℓ'.

(a) Prove that if d is a greatest common divisor of a and b, then $\frac{ab}{d}$ is a least common multiple of a and b.

(b) Let F be a field. Show that any two nonzero polynomials $p(x), q(x) \in F[x]$ have a least common multiple $m(x)$. Prove that $(p(x)) \cap (q(x))$ is the principal ideal generated by $m(x)$.

3 Consider the polynomials

$$f(x) = x^{10} + x^5 + 1 \qquad \text{and} \qquad g(x) = x^6 - 1$$

in $\mathbb{Q}[x]$.

(a) Use the Euclidean algorithm to find a greatest common divisor of $f(x)$ and $g(x)$. Write the greatest common divisor in the form

$$a(x)f(x) + b(x)g(x)$$

for some $a(x), b(x) \in \mathbb{Q}[x]$.

(You may omit a proof that the Euclidean algorithm is valid in $F[x]$. It is essentially identical to the proof used for $\mathbb{Z}$.)

(b) Find a least common multiple of $f(x)$ and $g(x)$.

(c) Describe the ideals $(x^{10} + x^5 + 1,\ x^6 - 1)$ and $(x^{10} + x^5 + 1) \cap (x^6 - 1)$.

4 Let F be a field, and let $R \subseteq F[x]$ be the subset of all polynomials whose coefficient of x is equal to 0.

(a) Prove that R is a subring of $F[x]$ and that R is an integral domain.

(b) Show that the constant polynomial 1 is a greatest common divisor of x^2 and x^3 in R.

(c) Show that there is no element of R which is a greatest common divisor of x^5 and x^6.