1 The rational root test

(a) Let

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x] \qquad (a_n \neq 0),$$

and suppose $p(\alpha) = 0$ for some $\alpha \in \mathbb{Q}$. Prove that if $\alpha = \frac{r}{s}$ is in lowest terms (i.e. $r, s \in \mathbb{Z}$ are relatively prime), then r divides a_0 and s divides a_n.

(b) Use part (a) to quickly find a rational root of $9x^5 + 8x^3 + 2x^2 - x + 2$.

(c) Use part (a) to show that $2x^3 - 7x + 3$ is irreducible in $\mathbb{Q}[x]$.

2 Let p be a prime number. Show that the polynomial

$$x^n - p$$

is irreducible over $\mathbb{Q}$ for any integer $n \geq 1$.

3 Show that the polynomial

$$(x-1)(x-2)\cdots(x-n)-1$$

is irreducible over $\mathbb{Q}$ for any integer $n \geq 1$. [HINT: Suppose it factors as $f(x)g(x)$ and consider the polynomial $f(x)+g(x)$ at the values $x = 1, \ldots, n$.]

4 Let n be a positive integer. Prove that the polynomial

$$1 + x + x^2 + \cdots + x^{n-1}$$

is irreducible in $\mathbb{Q}[x]$ if and only if n is prime.