1 Consider the polynomial $f(x) = 1 - x + x^2 - x^3 + x^4 \in \mathbb{Q}[x]$.

(a) Prove that $f(x)$ is irreducible over $\mathbb{Q}$. [HINT: $f(x) = \Phi_5(-x)$]

(b) Let α be a root of $f(x)$ in an extension field, and consider the simple extension $\mathbb{Q}(\alpha)$. What is the dimension of $\mathbb{Q}(\alpha)$ as a vector space over $\mathbb{Q}$? Use the element α to write down a basis of $\mathbb{Q}(\alpha)$ as a $\mathbb{Q}$-vector space.

(c) Write down a multiplication table for the elements of your basis from part (b).

(d) Use the multiplication table from part (c) to write the following elements as $\mathbb{Q}$-linear combinations of basis elements:

i. $(1+\alpha^2)^3$

ii. α^{-1}

iii. $\frac{\alpha^3+\alpha}{3+\alpha^2}$

2 Consider the polynomial $g(x) = x^3 + 2x + 2 \in \mathbb{F}_3[x]$.

(a) Prove that $g(x)$ is irreducible over $\mathbb{F}_3$.

(b) Let β be a root of $g(x)$ in an extension field, and consider the simple extension $\mathbb{F}_3(\beta)$. What is the dimension of $\mathbb{F}_3(\beta)$ as a vector space over $\mathbb{F}_3$? Use the element β to write down a basis of $\mathbb{F}_3(\beta)$ as an $\mathbb{F}_3$-vector space.

(c) Write down a multiplication table for the elements of your basis from part (b).

(d) Show that $g(x)$ factors in $\mathbb{F}_3(\beta)[x]$ as a product of 3 distinct linear factors.

(e) Show that $h(x) = x^3 + 2x + 1$ is also irreducible in $\mathbb{F}_3[x]$. Find a factorization of $h(x)$ as a product of linear factors in $\mathbb{F}_3(\beta)[x]$.

3 Suppose that E is an extension of a field F and that $[E : F]$ is a prime number. Show that E is a simple extension of F.

4 Let $\omega = e^{2\pi i/3} \in \mathbb{C}$ be a primitive third root of unity and let $\sqrt[3]{2}$ be the real cube root of 2. Compute $[\mathbb{Q}(\omega, \sqrt[3]{2}) : \mathbb{Q}]$.