1 Let F be a field and let α be algebraic over F. Prove that if $E \supseteq F$ is any field extension and

$$\varphi\colon F(\alpha) \to E$$

is a homomorphism such that $\varphi(a) = a$ for every $a \in F$, then φ is completely determined by the element $\varphi(\alpha) \in E$.

2 Let $E \supseteq F$ be a field extension, and let $p(x) \in F[x]$ be irreducible over F. Let $S \subseteq E$ be the (finite) set of all roots of $p(x)$ which are elements of E.
Show that if $\sigma \in \mathrm{Aut}(E/F)$ and $\alpha \in S$, then $\sigma(\alpha) \in S$. Conclude that there is a group action of $\mathrm{Aut}(E/F)$ on S given by

$$\sigma \cdot \alpha = \sigma(\alpha).$$

3 Let $G = \text{Aut}(\mathbb{Q}(\sqrt{5}, \sqrt{7})/\mathbb{Q})$.

(a) Compute $[\mathbb{Q}(\sqrt{5}, \sqrt{7}) : \mathbb{Q}]$ and give a basis for $\mathbb{Q}(\sqrt{5}, \sqrt{7})$ as a vector space over $\mathbb{Q}$.

(b) If $\sigma \in G$, then what are the possible values of $\sigma(\sqrt{5})$? What are the possible values of $\sigma(\sqrt{7})$?

(c) Completely describe the group G.