1 The formal derivative. Let F be a field, and let

$$D\colon F[x] \to F[x]$$

be the formal differentiation map. That is,

$$D(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1}.$$

We call this *formal* differentiation because we are just following the power rule here; there is no need to invoke (or even define) limits!

(a) Show that D is a homomorphism from the additive group $(F[x], +)$ to itself.

(b) Suppose F has characteristic zero. Find $\ker D$ and $\operatorname{im} D$.

(c) Suppose F has characteristic p, where p is a prime. Find $\ker D$ and $\operatorname{im} D$.

(d) Show that D is *not* a ring homomorphism.

2 Let F be a field, and let $D\colon F[x] \to F[x]$ be the formal differentiation map as in the previous problem. For $f(x) \in F[x]$, we let $f'(x) = D(f(x))$.

(a) Prove that
$$D(af(x)) = af'(x)$$
for all $a \in F$ and $f(x) \in F[x]$.

(b) Prove the **product rule**: If $f(x), g(x) \in F[x]$, then
$$D\big(f(x)g(x)\big) = f(x)g'(x) + f'(x)g(x).$$
[HINT: Proceed by induction on the degree of $f(x)g(x)$.]

(c) Let k be a positive integer. Prove that
$$D(f(x)^k) = k\,f(x)^{k-1}f'(x)$$
for all $f(x) \in F[x]$.

3 Let F be a field, $p(x) \in F[x]$ a polynomial, and $E \supseteq F$ a splitting field extension for $p(x)$ over F. We say that $\alpha \in E$ is a **repeated root** of $p(x)$ if $(x-\alpha)^k$ divides $p(x)$ for some integer $k > 1$. (The largest k such that $(x-\alpha)^k$ divides $p(x)$ is called the **multiplicity** of the root α.)

(a) Prove that α is a repeated root of $p(x)$ if and only if α is a root of $p(x)$ and α is also a root of $p'(x)$.

(b) Prove that α is a repeated root of $p(x)$ if and only if the minimal polynomial of α over F is a common factor of $p(x)$ and $p'(x)$.