THE GEOMETRY OF MATROIDS
LECTURE 1 EXERCISES

1. **The rank function**
   Let M be a matroid on ground set E. Define the **rank function** of M to be the function
   $$\begin{aligned} \mathrm{rk}_M \colon 2^E &\to \mathbb{Z} \\ S &\mapsto \max_{\substack{I \in \mathcal{I}(M) \\ \text{s.t. } I \subseteq S}} |I|; \end{aligned}$$
   That is, $\mathrm{rk}_M(S)$ is the cardinality of the largest independent subset of S.
   (a) Check that $\mathrm{rk}_M(E)$ is equal to $\mathrm{rk}(M)$, the rank of M.
   (b) Show that
   $$\mathrm{rk}_M(S) = \max_{B \in \mathcal{B}(M)} |B \cap S|,$$
   where the maximum is taken over all bases B of M.
   (c) Let $\mathcal{A}$ be a vector configuration. Describe the rank function $\mathrm{rk}_{M(\mathcal{A})}$.
   (d) Let G be a graph. Describe the rank function $\mathrm{rk}_{M(G)}$.

2. **Graphic and representable matroids**
   We say that a matroid M is
   - **graphic** if $M \cong M(G)$ for some graph G;
   - K**-representable** if $M \cong M(\mathcal{A})$ for some configuration $\mathcal{A}$ in a vector space over the field K;
   - **representable** if M is K-representable for some field K;
   - **regular** if M is K-representable for every field K.

   Prove that every graphic matroid is regular.

3. $\star$**The basis axioms**
   (a) Let M be a matroid on E, and let $\mathcal{B} = \mathcal{B}(M)$ be its set of bases. Prove that $\mathcal{B}$ satisfies the following:
   (B1) $\mathcal{B}$ is nonempty.
   (B2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.
   (b) Conversely, suppose that $\mathcal{B} \subseteq 2^E$ is a collection of subsets satisfying conditions **(B1)** and **(B2)**. Prove that there is a matroid M on ground set E such that $\mathcal{B}$ is the set of bases of M.