THE GEOMETRY OF MATROIDS
LECTURE 2 EXERCISES

1. **Independent sets from circuits**
   Let E be a finite set. Let $\mathcal{C} \subseteq 2^E$ be a collection of subsets of E satisfying conditions **(C1)**–**(C3)**, and define
   $$\mathcal{I} = \{I \subseteq E \mid \text{no subset of } I \text{ is in } \mathcal{C}\}.$$
   Prove that $C \subseteq E$ is a minimal subset of E not contained in $\mathcal{I}$ if and only if $C \in \mathcal{C}$.

2. **Independent sets, bases, and circuits via the rank function**
   Let M be a matroid on ground set E and let $S \subseteq E$. Recall that the rank of S, denoted $\mathrm{rk}_M(S)$, is the cardinality of the largest independent subset of S.
   (a) Prove that S is independent if and only if $\mathrm{rk}_M(S) = |S|$.
   (b) Prove that S is a basis if and only if
   $$\mathrm{rk}_M(S) = |S| = \mathrm{rk}(M).$$
   [Recall that $\mathrm{rk}(M) = \mathrm{rk}_M(E)$.]
   (c) Prove that S is a circuit if and only if S is nonempty and
   $$\mathrm{rk}_M(S) = \mathrm{rk}_M(S \setminus e) = |S| - 1$$
   for every $e \in S$.

3. $\star$**Fundamental circuits**
   Let M be a matroid on ground set E.
   (a) Show that for each basis B of M and element $e \in E \setminus B$, there is a unique circuit contained in $B \cup \{e\}$ and that this circuit must contain e. This circuit, denoted $C(e, B)$, is the **fundamental circuit of** e **with respect to** B.
   (b) Let C be a circuit of M and $e \in C$. Show that there exists a basis B of M such that $C = C(e, B)$; that is, every circuit can be expressed as a fundamental circuit.
   (c) Let B be a basis of M, $e \in E \setminus B$, and $f \in B$. Prove that $f \in C(e, B)$ if and only if $(B \setminus \{f\}) \cup \{e\}$ is a basis of M.