THE GEOMETRY OF MATROIDS
LECTURE 4 EXERCISES

1. **Flats of graphic matroids**
   (a) Let C_n be the cycle graph with n edges ($n \geq 3$). Describe the flats of $M(C_n)$.
   (b) Let K_n be the complete graph on n vertices. Describe the flats of $M(K_n)$.
   (c) Let G be an arbitrary graph. Describe the flats of $M(G)$.

2. **Spanning sets**
   Let M be a matroid on ground set E.
   (a) Show that $S \subseteq E$ is a spanning set of M if and only if $\text{rk}_M(S) = \text{rk}(M)$.
   (b) Show that $B \subseteq E$ is a basis of M if and only if B is both independent and spanning.
   (c) Show that $B \subseteq E$ is a basis of M if and only if B is a minimal spanning set.
   (d) Show that $H \subseteq E$ is a hyperplane of M if and only if H is a maximal non-spanning set.

3. $\star$**Circuits and closure**
   Let M be a matroid on ground set E.
   (a) Show that C is a circuit of M if and only if C is a minimal non-empty set with the property that $e \in \text{cl}(C \setminus e)$ for all $e \in C$.
   (b) Let $X \subseteq E$ be any subset. Prove that
   $$\text{cl}(X) = X \cup \{e \in E \mid \text{there exists } C \in \mathcal{C}(M) \text{ such that } e \in C \subseteq X \cup e\}.$$
   (c) Conclude that $F \subseteq E$ is a flat of M if and only if $|C \setminus F| \neq 1$ for every circuit $C \in \mathcal{C}(M)$.

4. **Uniform matroids**
   Let $U_{r,n}$ be the **uniform matroid of rank r on n elements**, which has ground set
   $$E(U_{r,n}) = [n] := \{1, 2, \ldots, n\}$$
   and independent sets
   $$\mathcal{I}(U_{r,n}) = \{I \subseteq [n] \mid |I| \leq r\}.$$
   Find the bases, circuits, rank function, closure operator, and flats of $U_{r,n}$.