THE GEOMETRY OF MATROIDS
LECTURE 5 EXERCISES

1. **Uniform matroids**
   Let $r \leq n$ be non-negative integers. Let $U_{r,n}$ be the **uniform matroid of rank** r **on** n **elements**, which has ground set
   $$E(U_{r,n}) = [n] := \{1, 2, \dots, n\}$$
   and independent sets
   $$\mathcal{I}(U_{r,n}) = \{I \subseteq [n] \mid |I| \leq r\}.$$
   Find the bases, circuits, rank function, closure operator, and flats of $U_{r,n}$.

2. **Are uniform matroids graphic?**
   (a) Show that $U_{2,4}$ is not graphic.
   (b) Find all pairs (r, n), $0 \leq r \leq n$, such that $U_{r,n}$ is graphic.

3. **Are uniform matroids representable?**
   (a) Prove that $U_{2,4}$ is representable over a field K if and only if $|K| \geq 3$.
   (b) Fix $n \geq 2$. Find necessary and sufficient conditions on the field K for $U_{2,n}$ to be K-representable.
   (c) Prove that a uniform matroid is representable any infinite field.
   (d) For each $0 \leq r \leq n$, explicitly determine the fields K over which $U_{r,n}$ is representable. Publish your results.

4. $\star$**Loops**
   Let M be a matroid on ground set E, and let $e \in E$. Prove the equivalence of the following statements.
   (a) $\{e\}$ is a circuit.
   (b) e is not contained in any independent set.
   (c) e is not contained in any basis.
   (d) $\mathrm{rk}(\{e\}) = 0$.
   (e) $e \in \mathrm{cl}(\emptyset)$.
   (f) $e \in \mathrm{cl}(X)$ for every $X \subseteq E$.
   (g) e is contained in every flat.
   We say that e is a **loop** of M if it satisfies any of the equivalent conditions (a)–(g).