THE GEOMETRY OF MATROIDS
LECTURE 7 EXERCISES

1. **Parallel elements**
   Let M be a matroid on ground set E, and let $e, f \in E$. We say that e and f are **parallel** in M if $e \neq f$ and $\{e, f\}$ is a circuit.
   (a) Prove that distinct elements e and f are parallel in M if and only if e and f are not loops and for each basis B containing e, $(B \setminus e) \cup f$ is also a basis.
   (b) Define a relation $\sim$ on E by
   $$e \sim f \qquad \Longleftrightarrow \qquad e = f \text{ or } e \text{ and } f \text{ are parallel in } M.$$
   Prove that $\sim$ is an equivalence relation on E. An equivalence class of this relation is called a **parallel class** of M.

2. **Coloops**
   Let M be a matroid on ground set E, and let $e \in E$. Prove the equivalence of the following statements.
   (a) e is not contained in any circuit.
   (b) For every independent set I, $I \cup e$ is independent.
   (c) e is contained in every basis.
   (d) If $X \subseteq E$ is any subset with $e \notin X$, then $\operatorname{rk}(X \cup e) = \operatorname{rk}(X) + 1$.
   (e) $\operatorname{cl}(E \setminus e) = E \setminus e$.
   (f) If $X \subseteq E$ is any subset with $e \notin X$, then $e \notin \operatorname{cl}(X)$.
   (g) For every flat F, $F \setminus e$ is a flat.
   We say that e is a **coloop** (or **isthmus**) of M if it satisfies any of the equivalent conditions (a)–(g).

3. $\star$**Circuit-hyperplane relaxation**
   Let M be a matroid on E, and suppose $H \subseteq E$ is a circuit of M which is also a hyperplane.
   (a) Prove that $\mathcal{B}' = \mathcal{B}(M) \cup \{H\}$ is the set of bases of a matroid M' on E. The matroid M' is called a **relaxation** of M.
   (b) Identify the circuits of M'.
   (c) Prove that every relaxation of the Fano matroid F_7 is isomorphic to the non-Fano matroid F_7^-.