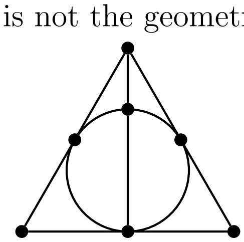
THE GEOMETRY OF MATROIDS
LECTURE 8 EXERCISES

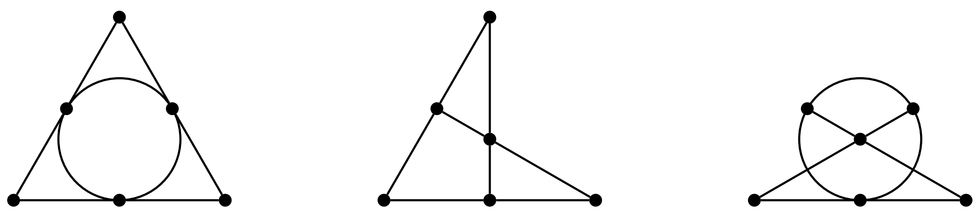
1. **Representability of F_7 and F_7^-**
   Let K be any field.
   (a) Let $\mathcal{A}$ be the configuration in K^3 consisting of the seven column vectors of the matrix
   $$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$
   Prove that $M(\mathcal{A}) \cong F_7$ if $\operatorname{char} K = 2$ and $M(\mathcal{A}) \cong F_7^-$ if $\operatorname{char} K \neq 2$.
   (b) Conversely, suppose that $\mathcal{A} = \{v_1, \ldots, v_7\}$ is a configuration in a K-vector space V such that $M(\mathcal{A}) \cong F_7$ (we may assume $\dim V = 3$). Prove that $\operatorname{char} K = 2$.
   HINTS:
   - By definition of F_7, the vectors in $\mathcal{A}$ are pairwise linearly independent, and the span of any two distinct vectors in $\mathcal{A}$ contains a unique third vector in $\mathcal{A}$.
   - We may freely replace any vector in $\mathcal{A}$ by a nonzero scalar multiple.
   - Without loss of generality, $\{v_1, v_2, v_3\}$ is a basis of V. We may choose coordinates so that $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, and $v_3 = (0,0,1)$.
   - Now, show that we may assume the vectors in $\mathcal{A}$ are the columns of the matrix
   $$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & \alpha & 0 & 1 & \delta \\ 0 & 0 & 1 & 0 & \beta & \gamma & \gamma\delta \end{bmatrix},$$
   where $\alpha, \beta, \gamma, \delta \in K$ are nonzero constants.
   - Show that $\beta = \gamma\delta$, $\delta = \alpha$, and $\beta + \alpha\gamma = 0$.

   (c) Modify the argument from (b) to show that if F_7^- is K-representable, then $\operatorname{char} K \neq 2$.

2. **The Deathly Hallows "matroid"**
   Show that the following diagram is not the geometric representation of a matroid.



3. $\star$**Geometric representation of** $M(K_4)$
Let F_7 be the Fano matroid, and let e be any element of its ground set. By removing the point e from the geometric representation of F_7, we obtain a new matroid which we shall denote $F_7 \setminus e$ (called the **deletion** of e from F_7).
   (a) Verify that, depending on the choice of e, the geometric representation of $F_7 \setminus e$ is one of the following.



   (b) Prove that the three geometric representations in part (a) in fact define isomorphic matroids.
   (c) Show that $F_7 \setminus e \cong M(K_4)$ for each e in the ground set of F_7.