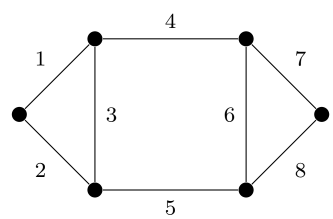
THE GEOMETRY OF MATROIDS
LECTURE 9 EXERCISES

1. **Graph connectivity is not matroidal**
   Let H_1 and H_2 be distinct connected components of a finite graph G, and let v_i be any vertex in H_i ($i = 1, 2$). Let G' be the graph obtained from G by fusing the vertices v_1 and v_2 into a single vertex v. Prove that $M(G) = M(G')$. Conclude that every graphic matroid is the matroid of a connected graph.

2. **Planar duality and matroid duality**
   Let G be the graph on edge set $E(G) = [8] = \{1, \ldots, 8\}$ pictured below.



   (a) Draw the planar dual G^* of G. Observe that there is a natural identification of the edge set $E(G^*)$ with $E(G) = [8]$.
   (b) Convince yourself that $M(G^*) = M(G)^*$.
   (c) Observe that $M(G^*)$ has three circuits of cardinality 2, eight circuits of cardinality 3, and four circuits of cardinality 4. Using the identification of $E(G^*)$ with $E(G)$, characterize the circuits of $M(G^*)$ in terms of the graph G.

3. $\star$**Cutsets**
   Let G be a finite graph. Given a set of edges $X \subseteq E(G)$, let $G \setminus X$ be the subgraph of G obtained by deleting all of the edges in X. We say that X is a **cutset** of G if X is a minimal set with the property that $G \setminus X$ has more connected components than G.
   (a) Prove that X is a cutset of G if and only if the complement $E(G) \setminus X$ is a hyperplane of the graphic matroid $M(G)$.
   (b) Prove that X is a cutset of G if and only if it is a minimal subset of $E(G)$ which has a nonempty intersection with every spanning forest.
   (c) Let $\mathcal{C}^*(G)$ denote the collection of all cutsets of G. Prove that $\mathcal{C}^*(G)$ satisfies the circuit axioms **(C1)**, **(C2)**, and **(C1)**. That is, $\mathcal{C}^*(G)$ is the collection of circuits of a matroid on $E(G)$.