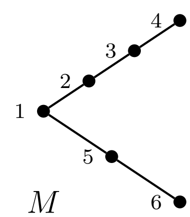
THE GEOMETRY OF MATROIDS
LECTURE 10 EXERCISES

1. ⋆**Duality and relaxation**
Let M be a matroid on E. As we proved in class today, a subset $H \subseteq E$ is a circuit-hyperplane of M (that is, $H \in \mathcal{C}(M)$ and $H \in \mathcal{H}(M)$) if and only if $E \setminus H$ is a circuit-hyperplane of M^*.
   (a) Prove that if M' is obtained from M by relaxing a circuit-hyperplane H (cf. Lecture 7 Exercise 3), then $(M')^*$ is obtained from M^* by relaxing $E \setminus H$.
   (b) Let M be the matroid pictured below.



   Draw a geometric representation of M^*. Is M^* simple?
   (c) Let M be as in part (b). Illustrate the result from part (a) by drawing geometric representations of M' and $(M')^*$.

2. ⋆**Fundamental cocircuits**
Let M be a matroid on E. If B is a basis of M and $e \in B$, then in M^* we have the fundamental circuit $C(e, E \setminus B) \in \mathcal{C}(M^*) = \mathcal{C}^*(M)$. (c.f. Lecture 2 Exercise 3). Considered as a cocircuit of M, this is called the **fundamental cocircuit** of e with respect to B and is denoted $C^*(e, B)$.
   (a) Show that $C^*(e, B)$ is the unique cocircuit of M that is disjoint from $B \setminus e$.
   (b) For $f \in E \setminus B$, show that $f \in C^*(e, B)$ if and only if $e \in C(f, B)$.
   (c) Let $C_1^*, \ldots, C_r^*$ be distinct cocircuits of a rank-r matroid M. Prove that the following statements are equivalent.
      (i) For each $j = 1, \ldots, r$, the cocircuit C_j^* is not contained in $\bigcup_{i \neq j} C_i^*$.
      (ii) There is a basis B of M such that $C_1^*, \ldots, C_r^*$ is a complete list of fundamental cocircuits with respect to B.