THE GEOMETRY OF MATROIDS
LECTURE 11 EXERCISES

1. **The dual closure operator**
   Let M be a matroid on E. Let $\mathrm{cl} = \mathrm{cl}_M \colon 2^E \to 2^E$ denote the closure operator of M and let $\mathrm{cl}^* = \mathrm{cl}_{M^*} \colon 2^E \to 2^E$ denote the closure operator of M^*.
   Given $X \subseteq E$ and $e \notin X$, let $Y = E \setminus (X \cup e)$. Prove that
   $$e \in \mathrm{cl}^*(X) \qquad \text{if and only if} \qquad e \notin \mathrm{cl}(Y).$$

2. **Some properties of flats**
   Let M be a matroid on E. Recall that $F \subseteq E$ is a **flat** of M if $\mathrm{cl}_M(F) = F$. Let $\mathcal{F}(M) \subseteq 2^E$ denote the collection of all flats of M.
   (a) Prove that M has a unique flat of rank 0, and that it is equal to
   $$\mathrm{cl}(\emptyset) = \{e \in E \mid e \text{ is a loop in } M\} = \bigcap_{F \in \mathcal{F}(M)} F.$$
   (b) Let $F = \{e_1, e_2 \ldots, e_t\}$ be a flat of M. Prove that
   $$F = \mathrm{cl}\left(\mathrm{cl}(e_1) \cup \mathrm{cl}(e_2) \cup \cdots \cup \mathrm{cl}(e_t)\right).$$
   (c) Let F and G be flats of M. We say that G **covers** F if G is a minimal flat properly containing F; that is, $F \subsetneq G$ and if $F \subseteq H \subseteq G$ for some flat H, then $H = F$ or $H = G$.
   Prove that if G covers F, then $\mathrm{rk}_M(G) = \mathrm{rk}_M(F) + 1$.
   (d) Suppose
   $$F = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k = G$$
   is a chain of flats ($F_i \in \mathcal{F}(M)$ for all i). Prove that $k \leq \mathrm{rk}(G) - \mathrm{rk}(F)$ with equality if and only if F_i covers F_{i-1} for all $i = 1, \ldots, k$.

3. $\star$**The flat axioms**
   Let E be a finite set. Let $\mathcal{F} \subseteq 2^E$ be a collection of subsets. Prove that $\mathcal{F}$ is the collection of flats of a matroid M on E if and only if $\mathcal{F}$ satisfies the following conditions:
   (F1) $E \in \mathcal{F}$.
   (F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.
   (F3) If $F \in \mathcal{F}$ and $\{G_1, \ldots, G_k\}$ is the set of minimal members of $\mathcal{F}$ properly containing F (that is, the G_i are all members of $\mathcal{F}$ which **cover** F, in the sense of Exercise 2 above), then $\{G_1 \setminus F, G_2 \setminus F, \ldots, G_k \setminus F\}$ is a partition of $E \setminus F$.