THE GEOMETRY OF MATROIDS
LECTURE 12 EXERCISES

1. **Hyperplane arrangements**
Let K be a field, and let $\mathcal{A} = \{v_e \mid e \in E\}$ be a configuration in a K-vector space V. Without loss of generality, we may assume the vectors in $\mathcal{A}$ span V.
For each $e \in E$, define the **hyperplane**
$$H_e = \{f \in V^* \mid f(v_e) = 0\}.$$
(a) Show that H_e is a codimension 1 linear subspace of V^* if v_e is a nonzero vector.
(b) Show that $H_e = V^*$ if $v_e = 0$. In this case, we call H_e a **degenerate hyperplane**.
(c) Show that $H_{e_1} = H_{e_2}$ if and only if $\operatorname{span}(v_{e_1}) = \operatorname{span}(v_{e_2})$ in V.
(d) Let $X \subseteq E$. Prove that X is independent in $M(\mathcal{A})$ if and only if
$$\operatorname{codim}\left(\bigcap_{e\in X} H_e\right) = |X|.$$

2. **Gale duality**
Let K be a field, and let $\mathcal{A} = \{v_e \mid e \in E\}$ be a configuration in a K-vector space V. Without loss of generality, we may assume the vectors in $\mathcal{A}$ span V.
Let $\varphi\colon K^E \to V$ be the surjective linear map which sends the standard basis vector $\delta_e \in K^E$ to v_e. If we let $W = \ker \varphi$, then we have the short exact sequence
$$0 \to W \xrightarrow{\iota} K^E \xrightarrow{\varphi} V \to 0.$$
The standard basis $\{\delta_e \mid e \in E\}$ allows us to identify $K^E \cong (K^E)^*$; let $\{\delta_e^* \mid e \in E\}$ be the dual basis. Thus, the dualized short exact sequence
$$0 \leftarrow W^* \xleftarrow{\iota^*} (K^E)^* \xleftarrow{\varphi^*} V^* \leftarrow 0$$
defines a configuration $\mathcal{A}' = \{g_e \mid e \in E\}$ in W^*, where $g_e = \iota^*(\delta_e^*)$. The configuration $\mathcal{A}'$ is called the **Gale dual** of $\mathcal{A}$.
(a) Check that $(\mathcal{A}')' = \mathcal{A}$.
(b) Let $X \subseteq E$. Prove that $\{v_e \mid e \in X\}$ is linearly independent in V if and only if $\{g_e \mid e \in E \setminus X\}$ spans W^*.
(c) Let $X \subseteq E$. Prove that $\{v_e \mid e \in X\}$ is a basis of V if and only if $\{g_e \mid e \in E \setminus X\}$ is a basis of W^*.
(d) Conclude that $M(\mathcal{A})^* = M(\mathcal{A}')$.