THE GEOMETRY OF MATROIDS
LECTURE 15 EXERCISES

1. $\star$**Flats of the restriction**

   Let M be a matroid on E, and let $X \subseteq E$.

   (a) Show that for any $S \subseteq X$, the closure of S in $M|X$ is
   $$\mathrm{cl}_{M|X}(S) = \mathrm{cl}_M(S) \cap X.$$
   (b) Show that the flats of $M|X$ are precisely
   $$\mathcal{F}(M|X) = \{F \cap X \mid F \in \mathcal{F}(M)\}.$$
   (c) Suppose now that X is a flat of M. Show that the lattice of flats $\mathcal{L}(M|X)$ is isomorphic to the lattice
   $$\mathcal{L}(M)_{\leq X} = \{F \in \mathcal{L}(M) \mid F \subseteq X\}.$$
   (d) Show that $\mathcal{L}(M|X)$ need not be isomorphic to $\mathcal{L}(M)_{\leq X}$ if $X \notin \mathcal{F}(M)$.

2. $\star$**The rank function of the contraction**

   Let M be a matroid on E and let $T \subseteq E$.

   (a) Show that the following equalities hold for any $S \subseteq E \setminus T$.
      (i) $\mathrm{crk}_{M\setminus T}(S) = \mathrm{crk}_M(S) - \mathrm{crk}_M(E \setminus T)$
      (ii) $\mathrm{null}_{M\setminus T}(S) = \mathrm{null}_M(S)$
      (iii) $\mathrm{crk}_{M/T}(S) = \mathrm{crk}_M(S \cup T)$
      (iv) $\mathrm{null}_{M/T}(S) = \mathrm{null}_M(S \cup T) - \mathrm{null}_M(T)$

   (b) Use equation (iii) or equation (iv) to derive the formula for the rank function of M/T: For any $S \subseteq E \setminus T$,
   $$\mathrm{rk}_{M/T}(S) = \mathrm{rk}_M(S \cup T) - \mathrm{rk}_M(T).$$

3. $\star$**Flats of the contraction**

   Let M be a matroid on E and let $T \subseteq E$.

   (a) Show that for any $S \subseteq E \setminus T$, the closure of S in M/T is
   $$\mathrm{cl}_{M/T}(S) = \mathrm{cl}_M(S \cup T) \setminus T.$$
   (b) Show that the flats of M/T are precisely
   $$\mathcal{F}(M/T) = \{F \subseteq E \setminus T \mid F \cup T \in \mathcal{F}(M)\}.$$
   (c) Show that the lattice of flats $\mathcal{L}(M/T)$ is isomorphic to the lattice
   $$\mathcal{L}(M)_{\geq \mathrm{cl}(T)} = \{F \in \mathcal{L}(M) \mid F \supseteq \mathrm{cl}_M(T)\}.$$