THE GEOMETRY OF MATROIDS
LECTURE 19 EXERCISES

1. **Truncation**
   Let M be a matroid on ground set E of rank $\mathrm{rk}(M) \geq 1$. Recall that the **truncation** of M is the matroid $\mathrm{trunc}(M)$ on E with independent sets
   $$\mathcal{I}(\mathrm{trunc}(M)) = \{I \in \mathcal{I}(M) \mid |I| \leq \mathrm{rk}(M) - 1\}.$$
   (a) Show that
   $$\mathcal{B}(\mathrm{trunc}(M)) = \{I \in \mathcal{I}(M) \mid |I| = \mathrm{rk}(M) - 1\}.$$
   (b) Show that
   $$\mathcal{C}(\mathrm{trunc}(M)) = \{C \in \mathcal{C}(M) \mid |C| \leq \mathrm{rk}(M)\} \cup \mathcal{B}(M).$$
   (c) Let $X \subseteq E$. Show that
   $$\mathrm{rk}_{\mathrm{trunc}(M)}(X) = \min\{\mathrm{rk}_M(X), \mathrm{rk}(M) - 1\}.$$
   (d) Describe the lattice of flats of $\mathrm{trunc}(M)$.
   (e) Show that $\mathrm{trunc}(U_{r,n}) = U_{r-1,n}$ if $1 \leq r \leq n$.

2. **Some inequalities**
   Let M be a matroid on ground set E. Show that the following inequalties hold for any subset $T \subseteq E$.
   (a) $\mathrm{rk}(M\backslash T) \geq \mathrm{rk}(M/T)$
   (b) $\mathrm{rk}_M(T) + \mathrm{rk}_M(E\backslash T) \geq \mathrm{rk}(M)$
   (c) $\mathrm{rk}_M(T) + \mathrm{rk}_{M^*}(T) \geq |T|$

3. $\star$**Separators**
   Let M be a matroid on ground set E, and let $T \subseteq E$. Prove that the following statements are equivalent.
   (a) $M\backslash T = M/T$
   (b) $\mathrm{rk}(M\backslash T) \leq \mathrm{rk}(M/T)$
   (c) $\mathrm{rk}_M(T) + \mathrm{rk}_M(E\backslash T) = \mathrm{rk}(M)$
   (d) $\mathrm{rk}_M(T) + \mathrm{rk}_{M^*}(T) = |T|$
   (e) If $C \in \mathcal{C}(M)$ is a circuit, then either $C \subseteq T$ or $C \subseteq E \setminus T$.
   (f) T is a union of connected components of M.
   (g) $M = M|T \oplus M\backslash T$.