THE GEOMETRY OF MATROIDS
LECTURE 20 EXERCISES

1. **The characteristic polynomial doesn't detect parallel elements**
   Let M be a loopless matroid, and suppose that e and f are parallel in M. Show that $\chi_M(q) = \chi_{M\setminus e}(q)$. Conclude that $\chi_M(q) = \chi_{\widetilde{M}}(q)$ for any loopless matroid M.

2. **Higgs lift**
   Let M be a matroid on ground set E. The **Higgs lift** of M is the matroid
   $$\mathrm{lift}(M) = \big(\mathrm{trunc}(M^*)\big)^*.$$
   (a) Let $X \subseteq E$. Show that
   $$\mathrm{rk}_{\mathrm{lift}(M)}(X) = \min\{\mathrm{rk}_M(X)+1, |X|\} = \begin{cases} \mathrm{rk}_M(X) & \text{if } X \in \mathcal{I}(M), \\ \mathrm{rk}_M(X)+1 & \text{if } X \notin \mathcal{I}(M). \end{cases}$$
   (b) Show that
   $$\mathcal{I}(\mathrm{lift}(M)) = \{X \subseteq E \mid \mathrm{null}_M(X) \leq 1\}.$$
   (c) Show that the bases of $\mathrm{lift}(M)$ are the nullity-1 spanning sets in M, i.e.,
   $$\mathcal{B}(\mathrm{lift}(M)) = \{X \subseteq E \mid \mathrm{rk}_M(X) = \mathrm{rk}(M) \text{ and } \mathrm{null}_M(X) = 1\}.$$
   (d) Show that $\mathrm{lift}(U_{r,n}) = U_{r+1,n}$ if $0 \leq r \leq n-1$.

3. $\star$**Free extension**
   Let M be a matroid on E, and let $e \notin E$. The **free extension** of M by e is the matroid
   $$M + e = \mathrm{trunc}(M \oplus U_{1,1}).$$
   Geometrically, $M + e$ is obtained from M by "adding a point in general position."
   (a) Show that
   $$\mathcal{B}(M+e) = \mathcal{B}(M) \cup \{I \cup e \mid I \in \mathcal{I}(M) \text{ and } |I| = \mathrm{rk}(M)-1\}.$$
   (b) Let $X \subseteq E$. Show that $\mathrm{rk}_{M+e}(X) = \mathrm{rk}_M(X)$ and
   $$\mathrm{rk}_{M+e}(X \cup e) = \begin{cases} \mathrm{rk}_M(X)+1 & \text{if } \mathrm{rk}_M(X) < \mathrm{rk}(M), \\ \mathrm{rk}(M) & \text{if } \mathrm{rk}_M(X) = \mathrm{rk}(M). \end{cases}$$
   (c) Show that $(M+e)\setminus e = M$.
   (d) Show that $(M+e)/e = \mathrm{trunc}(M)$.