THE GEOMETRY OF MATROIDS
LECTURE 21 EXERCISES

1. **Characteristic polynomial computations**
   Compute the characteristic polynomials of the following matroids.
   (a) $M(K_4)$
   (b) $M(K_5)$
   (c) $M(G)$, where G is the graph obtained from K_4 by deleting any edge.
   (d) $U_{3,n}$ for $n \geq 3$
   (e) $U_{4,n}$ for $n \geq 4$.

2. $\star$**Positivity of the characteristic polynomial**
   Let M be a loopless matroid of rank r, and let w_i be the coefficient of q^{r-i} in the characteristic polynomial $\chi_M(q)$, so that
   $$\chi_M(q) = w_0 q^r + w_1 q^{r-1} + \cdots + w_{r-1} q + w_r.$$
   (a) Prove that $w_0 = 1$.
   (b) Prove that the coefficients w_i are nonzero and alternate in sign. That is,
   $$(-1)^i w_i > 0$$
   for all $0 \leq i \leq r$. Equivalently, the polynomial $(-1)^r \chi_M(-q)$ has strictly positive coefficients.
   [HINT: Use the deletion/contraction formula and induct on the size of the ground set.]

3. **Free coextension**
   Let M be a matroid on E, and let $e \notin E$. The **free coextension** of M by e is the matroid
   $$M \times e = (M^* + e)^* = \mathrm{lift}(M \oplus U_{0,1}).$$
   (a) Show that
   $$\mathcal{B}(M \times e) = \{B \cup e \mid B \in \mathcal{B}(M)\} \cup \{X \subseteq E \mid \mathrm{rk}_M(X) = \mathrm{rk}(M) \text{ and } \mathrm{null}_M(X) = 1\}.$$
   (b) Let $X \subseteq E$. Show that $\mathrm{rk}_{M\times e}(X \cup e) = \mathrm{rk}_M(X) + 1$ and
   $$\mathrm{rk}_{M\times e}(X) = \begin{cases} \mathrm{rk}_M(X) & \text{if } X \in \mathcal{I}(M), \\ \mathrm{rk}_M(X) + 1 & \text{if } X \notin \mathcal{I}(M). \end{cases}$$
   (c) Show that $(M \times e)/e = M$.
   (d) Show that $(M \times e)\backslash e = \mathrm{lift}(M)$.