THE GEOMETRY OF MATROIDS
LECTURE 24 EXERCISES

1. **Log-concavity implies unimodality**
   A finite sequence $(a_0, a_1, \ldots, a_n)$ of positive numbers is **unimodal** if there is some $0 \leq k \leq n$ such that
   $$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n.$$
   (a) Show that every log-concave sequence of positive numbers is unimodal.
   (b) Find an example of a unimodal sequence of positive numbers which is not log-concave.

2. $\star$**Acyclic orientations**
   Let G be a connected simple graph. An **orientation** of G is an assignment of a direction to each edge (i.e. the undirected edge $\{v, w\} \in E(G)$ becomes either $v \to w$ or $v \leftarrow w$). A **directed cycle** in the resulting directed graph is a path
   $$v_1 \to v_2 \to \cdots \to v_k \to v_1.$$
   An **acyclic orientation** of G is an orientation which results in no directed cycles.
   (a) Let T be a tree on n vertices. Show that there are 2^{n-1} acyclic orientations of T.
   (b) Let C_n be the cycle graph with n vertices ($n \geq 3$). Compute the number of acyclic orientations of C_n.
   (c) Let G be an arbitrary connected simple graph. Prove that the number of acyclic orientations of G is
   $$(-1)^{|V(G)|} \operatorname{chr}_G(-1).$$
   [HINT: Use the deletion-contraction formula for chr_G and induct on the number of edges.]

3. **Bad colorings**
Let G be a graph. Given a coloring κ of the vertices of G using q colors, we say that an edge $e = \{v, w\}$ is **monochromatic** if its endpoints v and w are the same color. We let
$$\mathrm{mono}(\kappa) = \{e \mid e \text{ is monochromatic in } \kappa\},$$
so that κ is a proper q-coloring if and only if $\mathrm{mono}(\kappa) = \emptyset$.

The **bad coloring polynomial** of G is
$$\widetilde{\mathrm{chr}}_G(q,t) = \sum_{\kappa} t^{|\mathrm{mono}(\kappa)|}$$
where the sum is taken over all colorings κ of the vertex set of G using q colors.

(a) Show that
  (i) $\widetilde{\mathrm{chr}}_G(q,0) = \mathrm{chr}_G(q)$.
  (ii) $\widetilde{\mathrm{chr}}_G(q,1) = q^{|V(G)|}$.
  (iii) $\widetilde{\mathrm{chr}}_G(1,t) = t^{|E(G)|}$.

(b) Let $e \in E(G)$ be an edge. Prove that
$$\widetilde{\mathrm{chr}}_G(q,t) = \begin{cases} t\,\widetilde{\mathrm{chr}}_{G\setminus e}(q,t) & \text{if } e \text{ is a loop;} \\ \widetilde{\mathrm{chr}}_{G\setminus e}(q,t) + (t-1)\,\widetilde{\mathrm{chr}}_{G/e}(q,t) & \text{otherwise.} \end{cases}$$

(c) Use the formula from part (b) to conclude that $\widetilde{\mathrm{chr}}_G(q,t) \in \mathbb{Z}[q,t]$ (i.e., the bad coloring polynomial is actually a polynomial).

(d) Compute the bad coloring polynomial of K_4.