THE GEOMETRY OF MATROIDS
LECTURE 26 EXERCISES

1. **Möbius function computations**
   For each matroid M below, compute $\mu_M(\emptyset, F)$ for every flat F. Use this to compute the characteristic polynomial $\chi_M(q)$. (Compare Lecture 21 Exercise 1.)
   (a) $M(K_4)$
   (b) $M(K_5)$
   (c) $M(G)$, where G is the graph obtained from K_4 by deleting any edge.
   (d) $U_{3,n}$ for $n \geq 3$
   (e) $U_{4,n}$ for $n \geq 4$.

2. $\star$**The incidence algebra of a poset**
   Let $\mathcal{P}$ be a finite poset. Define

$$\mathrm{Int}(\mathcal{P}) = \{(x, y) \mid x, y \in \mathcal{P} \text{ with } x \leq y\}$$

   to be the set of intervals in $\mathcal{P}$.

   Given a field K, let $I(\mathcal{P}, K)$ be the K-vector space of all functions

$$\alpha\colon \mathrm{Int}(\mathcal{P}) \to K.$$

   If $\alpha, \beta \in I(\mathcal{P}, K)$, we define their product $\alpha\beta \in I(\mathcal{P}, K)$ to be the convolution

$$\alpha\beta(x, y) = \sum_{x \leq z \leq y} \alpha(x, z)\beta(z, y).$$

   Under this convolution product, $I(\mathcal{P}, K)$ is an associative K-algebra, called the **incidence algebra** of $\mathcal{P}$ over K.
   (a) Check that $I(\mathcal{P}, K)$ has a two-sided identity δ, given by

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y; \\ 0 & \text{if } x \neq y. \end{cases}$$

   (b) Show that $\alpha \in I(\mathcal{P}, K)$ has a right inverse if and only if $\alpha(x, x) \neq 0$ for all $x \in \mathcal{P}$. [HINT: Compute the inverse explicitly.]
   (c) Show $\alpha \in I(\mathcal{P}, K)$ has a right inverse if and only if it has a left inverse if and only if it has a two-sided inverse.
   (d) Define $\zeta \in I(\mathcal{P}, K)$ by

$$\zeta(x, y) = 1 \quad \text{for all } (x, y) \in \mathrm{Int}(\mathcal{P}).$$

   Compute $\zeta^{-1} \in I(\mathcal{P}, K)$.