THE GEOMETRY OF MATROIDS
LECTURE 27 EXERCISES

For these problems, we fix a finite poset $\mathcal{P}$ and field K. Let $I(\mathcal{P}, K)$ be the incidence algebra of $\mathcal{P}$ over K (cf. Lecture 26 Exercise 2). Recall that $I(\mathcal{P}, K)$ has a two-sided identity, δ.

1. **Counting chains**
   Let $\zeta \in I(\mathcal{P}, K)$ be defined by
   $$\zeta(x, y) = 1 \quad \text{for all } (x, y) \in \text{Int}(\mathcal{P}).$$
   (a) Let $x \leq y$ in $\mathcal{P}$. Show that $\zeta^2(x, y)$ is equal to the cardinality of the set
   $$\{z \mid x \leq z \leq y\}.$$
   (b) Let $x \leq y$ in $\mathcal{P}$. Elements $s_0, s_1, \ldots, s_k \in \mathcal{P}$ form a **multichain** of length k from x to y if
   $$x = s_0 \leq s_1 \leq \cdots \leq s_k = y.$$
   Show that $\zeta^k(x, y)$ is equal to the number of length k multichains from x to y.
   (c) A **chain** is a multichain in which every containment is strict. Show that
   $$(\zeta - \delta)^k(x, y)$$
   is equal to the number of length k chains from x to y.

2. $\star$**Möbius inversion via algebra**
   Consider the K-vector space $K^{\mathcal{P}}$ of all functions $\mathcal{P} \to K$.
   (a) For $\alpha \in I(\mathcal{P}, K)$ and $f \in K^{\mathcal{P}}$, define $\alpha \cdot f \in K^{\mathcal{P}}$ via
   $$(\alpha \cdot f)(x) = \sum_{y \geq x} \alpha(x, y) f(y).$$
   Prove that this action makes $K^{\mathcal{P}}$ into a left $I(\mathcal{P}, K)$-module.
   (b) In Lecture 26 Exercise 3, you proved that $\zeta \in I(\mathcal{P}, K)$ is invertible. Thus,
   $$f = \zeta \cdot g \qquad \Longleftrightarrow \qquad \zeta^{-1} \cdot f = g.$$
   Explain why this statement "is" Möbius inversion.
   (c) Prove the "dual" form of Möbius inversion by a similar argument.
   [HINT: Show that $K^{\mathcal{P}}$ is also a *right* $I(\mathcal{P}, K)$-module.]