THE GEOMETRY OF MATROIDS
LECTURE 28 EXERCISES

1. **Independent flats**
   Let M be a simple matroid on E, and let F be a flat of M. Show that $\mu_M(\emptyset, F) = \pm 1$ if and only if F is independent.

2. $\star$**The Möbius algebra**
   Let $\mathcal{L}$ be a finite lattice and let K be a field. Let $A(\mathcal{L}, K)$ be the K-algera with basis $\{\epsilon_x \mid x \in \mathcal{L}\}$ and multiplication

   $$\epsilon_x \cdot \epsilon_y = \epsilon_{x \vee y}.$$

   The algebra $A(\mathcal{L}, K)$ is the **Möbius algebra** of $\mathcal{L}$ over K.

   Let $B(\mathcal{L}, K)$ be the K-algebra with basis $\{\sigma_x \mid x \in \mathcal{L}\}$ and multiplication

   $$\sigma_x \cdot \sigma_y = \delta_{xy} \sigma_x.$$

   That is, $B(\mathcal{L}, K) \cong \bigoplus_{x \in \mathcal{L}} K$ with coordinate-wise multiplication.

   While $A(\mathcal{L}, K)$ and $B(\mathcal{L}, K)$ are both $|\mathcal{L}|$-dimensional vector spaces over K, it appears that $A(\mathcal{L}, K)$ has a more interesting ring structure.

   (a) Find an injective K-algebra homomorphism $\varphi\colon A(\mathcal{L}, K) \to B(\mathcal{L}, K)$ (specify the image of each basis element ϵ_x).
   (b) Conclude that $A(\mathcal{L}, K) \cong B(\mathcal{L}, K)$ as K-algebras.
   (c) In each of the following cases, describe the inverse isomorphism. That is, determine $\varphi^{-1}(\sigma_x) \in A(\mathcal{L}, K)$ for each $x \in \mathcal{L}$.
      (i) $\mathcal{L} = \mathcal{L}(U_{2,2})$
      (ii) $\mathcal{L} = \mathcal{L}(U_{3,3})$
      (iii) $\mathcal{L} = \mathcal{L}(U_{3,4})$
   (d) Describe the inverse isomorphism φ^{-1} in general.

3. $\star$**Weisner's theorem via the Möbius algebra**
   Let M be a loopless matroid on ground set E, and let K be a field. Let $A(\mathcal{L}(M), K)$ be the Möbius algebra of $\mathcal{L}(M)$ over K.

   Let F be a nonempty flat. Compute the product $\epsilon_F \cdot \sigma_\emptyset \in A(\mathcal{L}(M), K)$ two ways, first by using the σ-basis and then again using the ϵ-basis. Use these computations to provide an alternate proof of Weisner's theorem.