# THE GEOMETRY OF MATROIDS
# LECTURE 29 EXERCISES

1. **Broken circuits**
   Let $M$ be a matroid on ground set $[n] = \{1, \dots, n\}$. A **broken circuit** of $M$ is a set of the form $C \setminus m$, where $C \in \mathcal{C}(M)$ is a circuit and $m$ is the minimal element in $C$ (with respect to the standard ordering of $[n]$). A subset $I \subseteq [n]$ is called a **no-broken-circuit-set** (or **nbc-set** for short) if $I$ contains no broken circuits.
   (a) Compute all broken circuits and nbc-sets of $U_{3,5}$.
   (b) Compute all broken circuits and nbc-sets of $M(K_4)$.
   (c) Show that any nbc-set $I$ is independent.
   (d) Suppose $B \in \mathcal{B}(M)$ is a basis which is also an nbc-set. Show that $1 \in B$.
   (e) Let $I = \{i_1, i_2, \dots, i_k\}$, where $i_1 < i_2 < \cdots < i_k$. Show that $I$ is an nbc-set if and only if $i_t$ is the minimal element of $\mathrm{cl}(\{i_t, i_{t+1}, \dots, i_k\})$ for all $1 \le t \le k$.

2. $\star$**Broken circuits and Whitney numbers of the first kind**
   Let $M$ be a matroid of rank $r$ on ground set $E$. The **Möbius number** of $M$ is
   $$\mu(M) := (-1)^r \chi_M(0) = (-1)^r \mu_M(\emptyset, E).$$
   That is, $\mu(M) = |w_r|$ is the $r$th unsigned Whitney number of the first kind.
   (a) Check that the Möbius number satisfies the following properties.
      (i) $\mu(U_{0,0}) = 1$.
      (ii) If $M$ has a loop, then $\mu(M) = 0$.
      (iii) If $e$ is a coloop, then $\mu(M) = \mu(M/e)$.
      (iv) If $e$ is neither a loop nor a coloop, then $\mu(M) = \mu(M \backslash e) + \mu(M/e)$.
   (b) Fix a total order on the ground set $E$ (equivalently, choose a bijection $E \cong [n]$). Let
   $$\mathcal{N}(M) = \{B \in \mathcal{B}(M) \mid B \text{ is an nbc-set}\}$$
   be the set of nbc-bases of $M$. Check that the following properties hold.
      (i) $|\mathcal{N}(U_{0,0})| = 1$.
      (ii) If $M$ has a loop, then $\emptyset$ is a broken circuit and therefore $|\mathcal{N}(M)| = 0$.
      (iii) If $e$ is a coloop, then $|\mathcal{N}(M)| = |\mathcal{N}(M/e)|$
      (iv) If $e$ is neither a loop nor a coloop, then $|\mathcal{N}(M)| = |\mathcal{N}(M \backslash e)| + |\mathcal{N}(M/e)|$.
   (c) Conclude that $\mu(M) = |\mathcal{N}(M)|$.
   (d) Let $F$ be a flat of $M$. Show that $(-1)^{\mathrm{rk}(F)} \mu_M(\emptyset, F)$ is equal to the number of nbc-sets $I$ in $M$ such that $\mathrm{cl}(I) = F$. [HINT: Apply part (c) to the restriction $M|F$.]
   (e) Conclude that $|w_k|$, the $k$th unsigned Whitney number of the first kind, is equal to the number of nbc-sets in $M$ of cardinality $k$.