THE GEOMETRY OF MATROIDS
LECTURE 30 EXERCISES

1. $\star$**The reduced characteristic polynomial of the truncation**
Let M be a loopless matroid of rank $r \geq 1$ on ground set E.
(a) Show that
$$\overline{\chi}_{\mathrm{trunc}(M)}(q) = \frac{\overline{\chi}_M(q) - \overline{\chi}_M(0)}{q}.$$
(b) As usual, write
$$\overline{\chi}_M(q) = \sum_{k=0}^{r-1} (-1)^k \mu^k \, q^{r-1-k}.$$
Use part (a) to express the coefficients of $\overline{\chi}_{\mathrm{trunc}(M)}$ in terms of the coefficients μ^k of $\overline{\chi}_M$.
(c) Interpret the result of part (b) as a statement about the number of initial descending flags of flats in $\mathrm{trunc}(M)$ as compared to M. Does this make sense?

2. **Some Orlik-Solomon algebras**
For each matroid M below, compute the rank of $\mathrm{OS}^k(M)$ for all $k \geq 0$.
(a) $U_{2,3}$
(b) $U_{2,4}$
(c) $U_{3,4}$
(d) $M(K_4)$
(e) F_7

3. $\star$**The Orlik-Solomon algebra detects dependence**
Let M be a loopless matroid of rank r on ground set $[n] = \{1, \ldots, n\}$.
(a) Let $C \in \mathcal{C}(M)$ be a circuit. Show that $x_C = 0$ in $\mathrm{OS}^\bullet(M)$.
(b) Let D be a dependent set of M. Show that $x_D = 0$ in $\mathrm{OS}^\bullet(M)$.
(c) Let D be a dependent set of M. Use the graded Leibniz rule
$$\partial(x_S \wedge x_T) = x_S \wedge \partial x_T + (-1)^{|S|} \partial x_S \wedge x_T$$
to show that $\partial x_D = 0$ in $\mathrm{OS}^\bullet(M)$.
(d) Conclude that $\mathrm{OS}^k(M) = 0$ if $k > r$.