THE GEOMETRY OF MATROIDS
LECTURE 31 EXERCISES

1. **The flat-grading on the Orlik-Solomon algebra**
Let M be a loopless matroid of rank r on ground set $[n] = \{1, \ldots, n\}$. Let
$$\Lambda^\bullet[\mathbf{x}] = \Lambda^\bullet[x_1, \ldots, x_n]$$
be the graded exterior algebra over $\mathbb{Z}$. Recall that for each $S = \{i_1, \ldots, i_k\} \subseteq [n]$ with $i_1 < \cdots < i_k$, we write $x_S = x_{i_1} \wedge \cdots \wedge x_{i_k}$.

For each flat $F \in \mathcal{L}(M)$, let $\Lambda^F[\mathbf{x}]$ be the subgroup of $\Lambda^\bullet[\mathbf{x}]$ generated by elements x_S such that $\mathrm{cl}(S) = F$. Then
$$\Lambda^\bullet[\mathbf{x}] = \bigoplus_{F \in \mathcal{L}(M)} \Lambda^F[\mathbf{x}].$$
If $y \in \Lambda^F[\mathbf{x}]$, then we say y is **flat-homogeneous** and has **flat-grade** F.
(a) Let $y \in \Lambda^F[\mathbf{x}]$ and $z \in \Lambda^G[\mathbf{x}]$ be flat-homogeneous elements. Show that the exterior product $y \wedge z$ is flat-homogeneous with flat-grade $F \vee G = \mathrm{cl}(F \cup G)$.
(b) Show that the Orlik-Solomon ideal
$$I_{\mathrm{OS}} = \big(\partial x_C \mid C \text{ is a circuit of } M\big)$$
is flat-homogeneous (i.e., each generator ∂x_C is flat-homogeneous).
(c) Conclude that $\mathrm{OS}^\bullet(M) = \Lambda^\bullet[\mathbf{x}]/I_{\mathrm{OS}}$ inherits the flat-grading from $\Lambda^\bullet[\mathbf{x}]$. Moreover, this refines the standard grading in that
$$\mathrm{OS}^k(M) = \bigoplus_{F \in \mathcal{L}(M)_k} \mathrm{OS}^F(M),$$
where the sum is over all flats F of rank k and $\mathrm{OS}^F(M)$ is the image of $\Lambda^F[\mathbf{x}]$ in $\mathrm{OS}^\bullet(M)$.

2. ⋆⋆**The nbc-basis of the Orlik-Solomon algebra**
Let M be a loopless matroid of rank r on ground set $[n] = \{1, \dots, n\}$. If $I \subseteq [n]$ is an nbc-set (cf. Lecture 29 Exercise 1), then we shall call the element $x_I \in \mathrm{OS}^\bullet(M)$ an **nbc-monomial**. We shall show that the nbc-monomials form a basis for $\mathrm{OS}^\bullet(M)$.

It will be convenient to put the glex (graded lexicographic) order on the collection of subsets of $[n]$. Given $S, T \subseteq [n]$, we write $S < T$ if either $|S| < |T|$ or $|S| = |T|$ and when we write the elements of S and T in increasing order,

$$S = \{i_1, \dots, i_k\},\ T = \{j_1, \dots, j_k\} \qquad \text{with} \qquad i_1 < \cdots < i_k \text{ and } j_1 < \cdots < j_k,$$

there is some index t such that $i_1 = j_1$, $i_2 = j_2$, $\dots$, $i_{t-1} = j_{t-1}$, and $i_t < j_t$. That is, we compare subsets in the glex order by first comparing cardinality, then comparing smallest elements, then comparing second smallest elements, and so on.

(a) Let I be an independent set, and suppose $C \setminus m$ is a broken circuit contained in I. Show that $\partial x_{I \cup m} = 0$ in $\mathrm{OS}^\bullet(M)$ and use this to express x_I as a sum of elements of the form x_J, where J is independent and $J < I$.
(b) Use part (a) to show that the nbc-monomials span $\mathrm{OS}^\bullet(M)$.
(c) Use the flat-grading on $\mathrm{OS}^\bullet(M)$ to show that any minimal linear relation among nbc-monomials must occur among nbc-monomials contained in $\mathrm{OS}^F(M)$ for some flat F.
(d) Fix a flat F of rank k. Suppose

$$\sum a_I x_I = 0$$

is a linear relation among distinct nbc-monomials in $\mathrm{OS}^F(M)$. Apply the derivation ∂ to obtain a linear relation in $\mathrm{OS}^{k-1}(M)$. Express this degree-$(k-1)$ relation as a sum over *distinct* nbc-monomials. [HINT: If I is an nbc-set with $\mathrm{cl}(I) = F$, then I must contain the minimal element $e \in F$ (why?). Use the graded Leibniz rule and pay attention to this minimal element.]
(e) Use part (d) and induction on k to show that there are no nontrivial relations among the nbc-monomials.
(f) Conclude that

$$\operatorname{rank} \mathrm{OS}^F(M) = (-1)^{\mathrm{rk}(F)} \mu_M(\emptyset, F) \qquad \text{and} \qquad \operatorname{rank} \mathrm{OS}^k(M) = |w_k|$$

(cf. Lecture 29 Exercise 2).