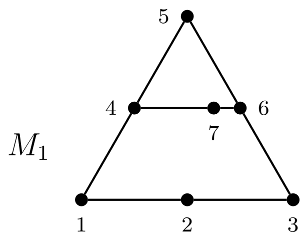
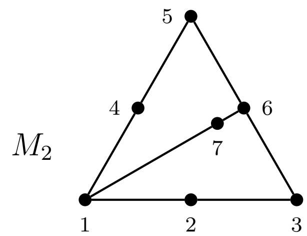
THE GEOMETRY OF MATROIDS
LECTURE 32 EXERCISES

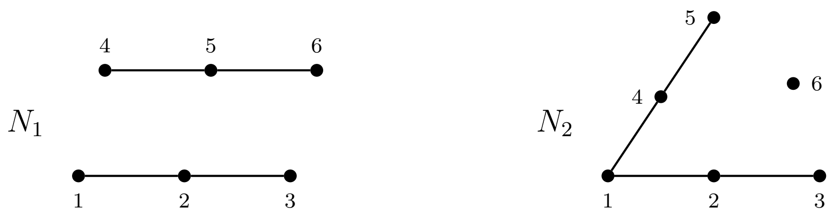
1. ⋆**Using the Orlik-Solomon algebra to distinguish matroids**
   Let M_1 and M_2 be the simple rank 3 matroids shown below.





   (a) Show that $\chi_{M_1}(q) = \chi_{M_2}(q)$. Conclude that $\operatorname{rank} \mathrm{OS}^k(M_1) = \operatorname{rank} \mathrm{OS}^k(M_2)$ for all $k \geq 0$.
   (b) Show that $\mathrm{OS}^\bullet(M_1) \ncong \mathrm{OS}^\bullet(M_2)$.

2. ⋆**The Orlik-Solomon algebra determines the matroid . . . or does it?** Let M be a loopless matroid.
   (a) Use Lecture 30 Exercise 3 and Lecture 31 Exercise 2 to show that a monomial x_I is nonzero in $\mathrm{OS}^\bullet(M)$ if and only if $I \in \mathcal{I}(M)$.
   (b) Let N_1 and N_2 be the simple rank 3 matroids shown below.



Show that the automorphism of $\Lambda^\bullet[x_1, \ldots, x_6]$ given by

$$\begin{aligned} x_1 &\mapsto x_1 \\ x_2 &\mapsto x_2 \\ x_3 &\mapsto x_3 \\ x_4 &\mapsto x_3 - x_5 + x_6 \\ x_5 &\mapsto x_4 - x_5 + x_6 \\ x_6 &\mapsto x_6 \end{aligned}$$

induces an isomorphism $\mathrm{OS}^\bullet(N_1) \cong \mathrm{OS}^\bullet(N_2)$.
   (c) Part (a) seems to say that the Orlik-Solomon algebra $\mathrm{OS}^\bullet(M)$ completely determines the matroid M, but in part (b) we found a pair of non-isomorphic matroids with isomorphic Orlik-Solomon algebras. How do you reconcile these seemingly contradictory observations?