THE GEOMETRY OF MATROIDS
LECTURE 35 EXERCISES

1. $\star$**The wonderful model of the braid arrangement**
   In this problem, we consider the matroid $M(K_4)$.
   (a) Prove that the configuration

   $$\mathcal{A} = \{(1,-1,0,0),(1,0,-1,0),(1,0,0,-1),(0,1,-1,0),(0,1,0,-1),(0,0,1,-1)\}$$

   of vectors in $\mathbb{C}^4$ is a representation of $M(K_4)$ over $\mathbb{C}$.
   (b) Show that the configuration from part (a) spans the subspace

   $$V = \{(v_1, v_2, v_3, v_4) \mid v_1 + v_2 + v_3 + v_4 = 0\} \subseteq \mathbb{C}^4.$$

   (c) Our choice of coordinates on $\mathbb{C}^4$ gives an isomorphism $(\mathbb{C}^4)^* \cong \mathbb{C}^4$. Check that

   $$V^{\perp} = \operatorname{span}\{(1,1,1,1)\} \subseteq \mathbb{C}^4.$$

   and

   $$V^* \cong \mathbb{C}^4/V^{\perp}.$$

   Conclude that if $x_1, \ldots, x_4$ are the coordinate functions on $\mathbb{C}^4$, then $x_i - x_j$ $(i \neq j)$ is a well-defined linear functional on V^*.
   (d) For $1 \leq i < j \leq 4$, let $H_{ij} \subseteq V^*$ be the hyperplane defined by $x_i - x_j = 0$. Check that

   $$\{H_{ij} \mid 1 \leq i < j \leq 4\}$$

   is the hyperplane arrangement in V^* associated to the configuration from part (a).
   (e) Draw a (cartoon) picture of the projectivized hyperplane arrangement

   $$\{\mathbb{P}H_{ij} \mid 1 \leq i < j \leq 4\}$$

   in $\mathbb{P}V^*$. (It may help to begin by choosing an isomorphism $V^* \cong \mathbb{C}^3$.)
   (f) Let $Y_{\mathcal{A}}$ be the wonderful compactification of the projective complement $\mathbb{P}U_{\mathcal{A}}$ of the hyperplane arrangement described in part (d). Describe the boundary divisor $\partial Y_{\mathcal{A}}$.
   More precisely, list the irreducible components of the boundary divisor and show that each is isomorphic to $\mathbb{P}^1$. Given two boundary components, D_F and D_G, when is the intersection $D_F \cap D_G$ nonempty?