THE GEOMETRY OF MATROIDS
LECTURE 36 EXERCISES

1. $\star$**Intersections of boundary components**
Let M be a $\mathbb{C}$-representable simple matroid on ground set E. Let $\mathcal{A}$ be a vector configuration in a $\mathbb{C}$-vector space V with $M = M(\mathcal{A})$, and let $U_{\mathcal{A}}$ be the complement of the associated hyperplane arrangement in V^*. For each non-empty proper flat $F \in \mathcal{L}(M)$, we have the boundary component D_F in the wonderful compactification $Y_{\mathcal{A}}$ of $\mathbb{P}U_{\mathcal{A}}$.
(a) Use the isomorphism
$$D_F \cong \mathbb{P}H_F \times \mathbb{P}(V^*/H_F)$$
described in class to prove that $D_F \cap D_G$ is nonempty if and only if $F \subseteq G$ or $G \subseteq F$.
(b) More generally, let
$$\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E$$
be a chain of proper, nonempty flats. Show that the intersection
$$D_{F_1} \cap \cdots \cap D_{F_k}$$
is nonempty and is isomorphic to a product of projective spaces.

2. $\star$**The Chow ring of** $U_{3,4}$
Consider the Chow ring $A^\bullet(U_{3,4})$, which is generated by $x_1, \ldots, x_4$ and $x_{12}, \ldots, x_{34}$. Show that the following equalities hold in $A^\bullet(U_{3,4})$.
(a) $x_i x_{ij} = x_1 x_{12}$ for all $1 \leq i, j \leq 4$.
(b) $x_i^2 = -2x_1 x_{12}$ for all $1 \leq i \leq 4$.
(c) $x_{ij}^2 = -x_1 x_{12}$ for all $1 \leq i, j \leq 4$.
Conclude that $A^2(U_{3,4})$ is isomorphic to either $\mathbb{Z}$ or $\{0\}$. Can you tell which it is?