# THE GEOMETRY OF MATROIDS
# LECTURE 37 EXERCISES

1. $\star$**Psi classes and square-free monomials**
Let $M$ be a simple matroid of rank $r$ on ground set $E$. For $i \in E$, define elements $\alpha, \beta \in A^1(M)$ by
$$\alpha = \sum_{G \ni i} x_G \qquad \text{and} \qquad \beta = \sum_{G \not\ni i} x_G.$$
It follows from the definition of $A^\bullet(M)$ that the elements $\alpha$ and $\beta$ are independent of the choice of $i$. (You should check this!)

Let $F \in \mathcal{L}(M) \setminus \{\emptyset, E\}$ be a nonempty proper flat. Define the **psi classes** $\psi_F^\pm \in A^1(M)$ by
$$\psi_F^- = \alpha - \sum_{G \supseteq F} x_G \qquad \text{and} \qquad \psi_F^+ = \beta - \sum_{G \subseteq F} x_G.$$
(a) Choose elements $i \in F$ and $j \notin F$. Show that
$$x_F = \sum_{G \ni j} x_G \; - \sum_{\substack{G \ni i \\ G \neq F}} x_G$$
as elements of $A^1(M)$.
(b) Use part (a) to show that
$$x_F^2 = -x_F(\psi_F^- + \psi_F^+)$$
in $A^2(M)$.
(c) Choose elements $i \in F$ and $j \notin F$. Use part (b) and the quadratic relations on $A^\bullet(M)$ to show
$$x_F^2 = -x_F \left( \sum_{\substack{G \subsetneq F \\ i \in G}} x_G \; + \sum_{\substack{G \supsetneq F \\ j \notin G}} x_G \right).$$
Conclude that the square-free monomials $x_{F_1} x_{F_2}$, where $F_1 \subsetneq F_2$, generate $A^2(M)$.
(d) Show that $A^\bullet(M)$ is generated by the square-free monomials $x_{F_1} \cdots x_{F_\ell}$, where
$$\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_\ell \subsetneq E.$$
HINT: Suppose we have a monomial $x_{F_1}^{d_1} \cdots x_{F_\ell}^{d_\ell}$, where each $d_k \geq 1$. If $d_k > 1$ for some $k$, then
$$x_{F_1}^{d_1} \cdots x_{F_\ell}^{d_\ell} = -\, x_{F_1}^{d_1} \cdots x_{F_k}^{d_k - 1} \cdots x_{F_\ell}^{d_\ell} \, (\psi_{F_k}^- + \psi_{F_k}^+).$$
Now, rewrite $(\psi_{F_k}^- + \psi_{F_k}^+)$ by choosing some $i \in F_k \setminus F_{k-1}$ and $j \in F_{k+1} \setminus F_k$.