MA 764: The Geometry of Matroids

www.ms.uky.edu/~mbku225/math764

Def: A <u>matroid</u> M is a pair $(E, \mathcal{I})$, where

- E is a finite set, the <u>ground set</u> of M
- $\mathcal{I} \subseteq 2^E$ is a collection of subsets of E, called <u>independent sets</u>, satisfying

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$. [Augmentation]

<u>Notes</u>:

- (I1) + (I2) imply $\mathcal{I}$ is an <u>abstract simplicial complex</u> on vertex set E.
- Subsets of E which are not in $\mathcal{I}$ are called <u>dependent</u>.
- Sometimes write $E(M)$ or $\mathcal{I}(M)$ to specify the matroid M.

Def: A basis of a matroid M is a maximal independent set.

Lemma: If B_1 and B_2 are bases of M, then $|B_1| = |B_2|$.

Proof: (I3)

Def: The rank of a matroid M, denoted $rk(M)$, is the cardinality of any basis.

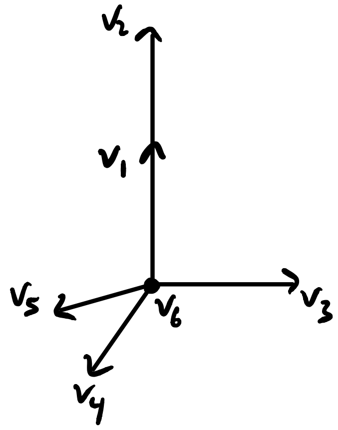
Main Examples

① Let V be a f.d. vector space over a field K, and let E be a finite set. A configuration $\mathcal{A}$ of vectors in V is a function (not nec. injective)

$$E \to V$$
$$e \mapsto v_e$$

(Equivalently, $\mathcal{A}$ is the (multi)set $\{v_e \mid e \in E\}$. If we order E then we can represent $\mathcal{A}$ as a matrix

$$\left[v_{e_1} \middle| v_{e_2} \middle| \cdots \middle| v_{e_n} \right]$$
)

Ex: $\mathcal{A} =$ 

$$\begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 0 & -1 & 0 \end{bmatrix}$$

A configuration $\mathcal{A}$ defines a matroid $M(\mathcal{A})$:

- $E(M(\mathcal{A})) = E$, the indexing set
- Independence is linear independence, i.e.

$$\mathcal{I}(M(\mathcal{A})) = \{ I \subseteq E \mid \{v_e \mid e \in I\} \text{ is a lin. indep. set} \}$$

In the example, the bases are

$$\{134, 135, 145, 234, 235, 245, 345\}$$

+ the independent sets are subsets of these. This matroid has rank 3.

loops + multiple edges allowed

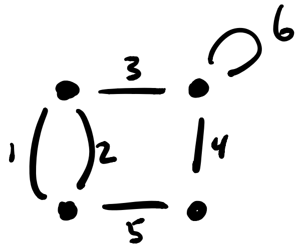
② A finite graph G defines a matroid $M(G)$:

- $E(M(G)) = E(G)$, the set of edges
- Independent sets = acyclic sets of edges, i.e.

$$\mathcal{I}(M(G)) = \{ I \subseteq E(G) \mid \text{no cycle can be formed from edges in } I \}$$

It follows that bases of $M(G)$ are spanning forests

Ex:

$G =$ 

$\to M(G)$ has bases $\{134, 135, 145, 234, 235, 245, 345\}$

The same matroid!

Def: If M_1 and M_2 are matroids, then they are <u>isomorphic</u> if there exists a bijection

$$f: E(M_1) \to E(M_2)$$

such that

$$I \in \mathcal{I}(M_1) \iff f(I) \in \mathcal{I}(M_2).$$

In this case, write $M_1 \cong M_2$.