Thm: Let E be a finite set. A collection $\mathcal{C} \subseteq 2^E$ of subsets of E is the set of circuits of a matroid on E if and only if $\mathcal{C}$ satisfies

(C1) $\emptyset \notin \mathcal{C}$.

(C2) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If $C_1, C_2 \in \mathcal{C}$ are distinct circuits and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ with $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.

Proof: ($\Rightarrow$) Last time

($\Leftarrow$) Conversely, suppose $\mathcal{C} \subseteq 2^E$ satisfies (C1)-(C3).

Claim: $\mathcal{I} = \{ I \subseteq E \mid \text{no subset of } I \text{ is in } \mathcal{C} \}$ is the collection of indep. sets of a matroid M.

Exercise 1 last time:

$C \in \mathcal{C} \iff C$ is a minimal set not contained in $\mathcal{I}$.

$\Longrightarrow \mathcal{C}$ will be the circuits of M.

We must show $\mathcal{I}$ satisfies (I1) - (I3).

(I1): The only subset of $\emptyset$ is itself, which is not in $\mathcal{C}$ by (C1).
So $\emptyset \in \mathcal{I}$.

$(I2)$: If $I \in \mathcal{I}$ and $J \subseteq I$, then subsets of J are also subsets of I, so $J \in \mathcal{I}$ also.

$(I3)$: Let $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$.

There exist subsets of $I_1 \cup I_2$ which are in $\mathcal{I}$ and larger than I_1 (e.g. I_2).

$(I3)$ is true for the pair (I_1, I_2)

$\iff$ such a set exists which contains I_1

Suppose not. Let J be such a set with $|I_1 \setminus J| > 0$ minimal.

Fix $e \in I_1 \setminus J$.

For <u>each</u> $f \in J \setminus I_1$, set

$$T_f = (J \setminus f) \cup e$$

Then $T_f \subseteq I_1 \cup I_2$ and $|I_1 \setminus T_f| < |I_1 \setminus J|$.

By minimality, $T_f \notin \mathcal{I}$.

$\Rightarrow T_f$ has a subset $C_f \in \mathcal{C}$.

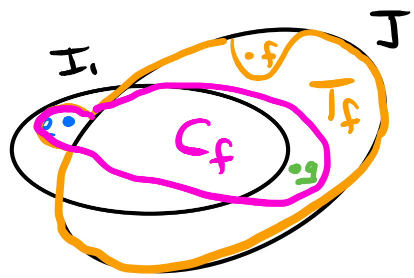
Observe: • $f \notin C_f$ (since $C_f \subseteq T_f$)

• $e \in C_f$ (otherwise, $C_f \subseteq J$

• C_f contains some element of $J \setminus I_1$

(otherwise $C_f \subseteq (I_1 \cap J) \cup e \subseteq I_1$)

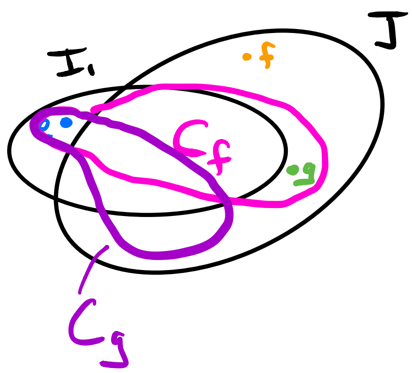
Picture:



Now fix $f \in J \setminus I_1$ and let g be an element in $J \setminus I_1$ with $g \in C_f$

Then $g \neq f$ (since $f \notin C_f$) and $C_g \in \mathcal{C}$ with $C_g \neq C_f$ (since $g \notin C_g$)

Picture:



So (C3) implies there is $C \in \mathcal{C}$ with

$$C \subseteq (C_f \cup C_g) \setminus e$$

But $C_f \cup C_g \subseteq J \cup e \Rightarrow C \subseteq J$,

a contradiction of $J \in \mathcal{I}$.

So (I3) holds. $\blacksquare$