More cryptomorphic definitions

The rank function (cf. Day 1 Exercise 1)

Def: Let M be a matroid on E. The *rank function* of M is

$$\mathrm{rk}_M : 2^E \longrightarrow \mathbb{Z}$$
$$X \longmapsto \max_{\substack{I \in \mathcal{I}(M) \\ I \subseteq X}} |I|$$

- $\mathrm{rk}_M(X)$ is the *rank of X*
- $\mathrm{rk}_M(E) = \mathrm{rk}(M)$
- Write rk instead of rk_M if M is clear
- $I \in \mathcal{I}(M) \iff \mathrm{rk}_M(I) = |I|$

Thm: Let E be a finite set, and $\mathrm{rk} : 2^E \to \mathbb{Z}$. Then $\mathrm{rk} = \mathrm{rk}_M$ for some matroid M on E if and only if it satisfies:

(R1) For all $X \subseteq E$, $0 \leq \mathrm{rk}(X) \leq |X|$.

(R2) If $X \subseteq Y \subseteq E$, then $\mathrm{rk}(X) \leq \mathrm{rk}(Y)$.

[Submodular Inequality] (R3) If $X, Y \subseteq E$, then

$$\mathrm{rk}(X \cup Y) + \mathrm{rk}(X \cap Y) \leq \mathrm{rk}(X) + \mathrm{rk}(Y).$$

Ex: If G is a finite graph and $X \subseteq E(G)$, then

$$\mathrm{rk}_{M(G)}(X) = \# V(G) - \#\{\text{conn. components of } G[X]\}$$

where $G[X]$ is the graph on vertices $V(G)$ and edges X.

Ex: If $\mathcal{A}$ is a vector configuration in a vector space V, indexed by E, then for $X \subseteq E$

$$\mathrm{rk}_{M(\mathcal{A})}(X) = \dim\left(\operatorname{span}\{v_e \mid e \in X\}\right)$$

Closure

Def: Let M be a matroid on E. The closure operator is the function $\mathrm{cl}_M : 2^E \longrightarrow 2^E$ defined by

$$\mathrm{cl}_M(X) = \{e \in E \mid \mathrm{rk}_M(X \cup e) = \mathrm{rk}_M(X)\}.$$

- $\mathrm{cl}_M(X)$ is the closure or span of X.
- Write cl for cl_M if M is clear.
- $I \in \mathcal{I}(M) \iff$ for every $e \in I$, $e \notin \mathrm{cl}(I \setminus e)$.

Lemma: $rk(cl(X)) = rk(X)$

Proof: Let $I \subseteq X$ be a maximal independent subset, so that $rk(X) = |I|$.

For every $e \in cl(X) \setminus X$,

$$rk(I \cup e) \underset{(R2)}{\leq} rk(X \cup e) \underset{\substack{\text{def.}\\ \text{of}\\ cl}}{=} rk(X) = |I| = rk(I) \underset{(R2)}{\leq} rk(I \cup e)$$

So we have equality throughout, and

$$rk(I \cup e) = rk(I) = |I| < |I \cup e|$$

Thus, $I \cup e$ is dependent

$\Rightarrow I$ is a maximal indep. subset of $cl(X)$

$\Rightarrow rk(cl(X)) = |I| = rk(X)$.

$\blacksquare$

Thm: Let E be a finite set and $cl: 2^E \to 2^E$.

Then $cl = cl_M$ for some matroid M on E if and only if

(CL1) For all $X \subseteq E$, $X \subseteq cl(X)$.

(CL2) If $X \subseteq Y \subseteq E$, then $cl(X) \subseteq cl(Y)$.

(CL3) For all $X \subseteq E$, $cl(cl(X)) = cl(X)$.

(CL4) If $X \subseteq E$ and $e, f \in E$, then

[MacLane - Steinitz Exchange]

$f \in cl(X \cup e) \setminus cl(X)$

implies $e \in cl(X \cup f)$.

<u>Def</u>: Let M be a matroid on E.

- $S \subseteq E$ is a <u>spanning set</u> of M if $cl(S) = E$.
- $F \subseteq E$ is a <u>closed set</u> or <u>flat</u> of M if $cl(F) = F$.

- A <u>point</u> of M is a flat of rank 1
- A <u>line</u> of M " " " rank 2
- A <u>plane</u> of M " " " rank 3

- A <u>copoint</u> or <u>hyperplane</u> " " " rank $rk(M) - 1$.