Simple Matroids

Def: Let M be a matroid on E.

- Wednesday, Exercise 4: $e \in E$ is a loop if $\{e\}$ is a circuit.
- Today, Exercise 1: $e, f \in E$ are parallel if $e \neq f$ and $\{e,f\}$ is a circuit.

A matroid is simple if it has no loops and no pairs of parallel edges.

Ex: $M(G)$ is simple $\Leftrightarrow$ G is a simple graph.

There is a natural simplification process

$$\text{matroid } M \rightsquigarrow \text{simple matroid } \tilde{M} = si(M)$$

such that $\tilde{M} = M \Leftrightarrow M$ is simple.

Informally, construct $\tilde{M}$ from M by deleting loops and collapsing parallel elements.

More precisely, the relation on $E = E(M)$

$$e \sim f \iff e = f \text{ or } e \text{ and } f \text{ are parallel}$$

is an equivalence relation (Ex. 1)

A non-loop equivalence class of $\sim$ is a <u>parallel class</u>.

Define $\widetilde{M}$ on ground set

$$\widetilde{E} = \{\text{parallel classes of } M\} = E \setminus \{\text{loops}\} / \sim$$

and with independence defined by taking representatives:

If $P_1, \dots, P_k \in \widetilde{E}$ are distinct parallel classes and $e_i \in P_i$, then

$$\{P_1, \dots, P_k\} \in \mathcal{I}(\widetilde{M}) \iff \{e_1, \dots, e_k\} \in \mathcal{I}(M).$$

<u>Check</u>: (I1) - (I3) hold.

<u>Ex 1(a)</u>: $e \sim f$ and $I \in \mathcal{I}(M)$ with $e \in I$

$$\implies (I \setminus e) \cup f \in \mathcal{I}(M)$$

<u>Prop</u>: If $P_1, \dots, P_k$ are distinct parallel classes, then

$$\{P_1, \dots, P_k\} \text{ is a flat of } \widetilde{M} \iff P_1 \sqcup \dots \sqcup P_k \sqcup \{\text{loops of } M\} \text{ is a flat of } M.$$

Representable matroids

Let $\mathcal{A} = \{v_e \mid e \in E\}$ be a configuration in a K-vector space V. Then $M(\mathcal{A})$ is simple if and only if

- $v_e \neq 0$ for all $e \in E$
- $e \neq f \Rightarrow \mathrm{span}(v_e) \neq \mathrm{span}(v_f)$.

If $M(\mathcal{A})$ is simple, then we may <u>projectivize</u> $\mathcal{A}$:

$$V \setminus \{0\} \longrightarrow \mathbb{P}V = \{\text{lines in } V\} = V\setminus\{0\} / K^\times$$

$$v \longmapsto [v] := \mathrm{span}(v)$$

<u>Main fact</u>: • If $W \subseteq V$ subspace def. by lin. forms $f_1, \ldots, f_k$, then $\mathbb{P}W \subseteq \mathbb{P}V$ is defined by the same forms.

- $\dim \mathbb{P}V = \dim V - 1$

  points in $\mathbb{P}V \leftrightarrow$ 1-dim subspaces of V

  lines in $\mathbb{P}V \leftrightarrow$ 2-dim subspaces of V

  etc.

If $M(\mathcal{A})$ is simple, define $\mathbb{P}\mathcal{A} = \{[v_e] \mid e \in E\}$.

Examples: $\mathcal{A}$ conf. in V s.t. $M(\mathcal{A})$ is simple.

WLOG assume $\mathcal{A}$ spans $V \Rightarrow rk(M(\mathcal{A})) = \dim V$
$= \dim \mathbb{P}V + 1$

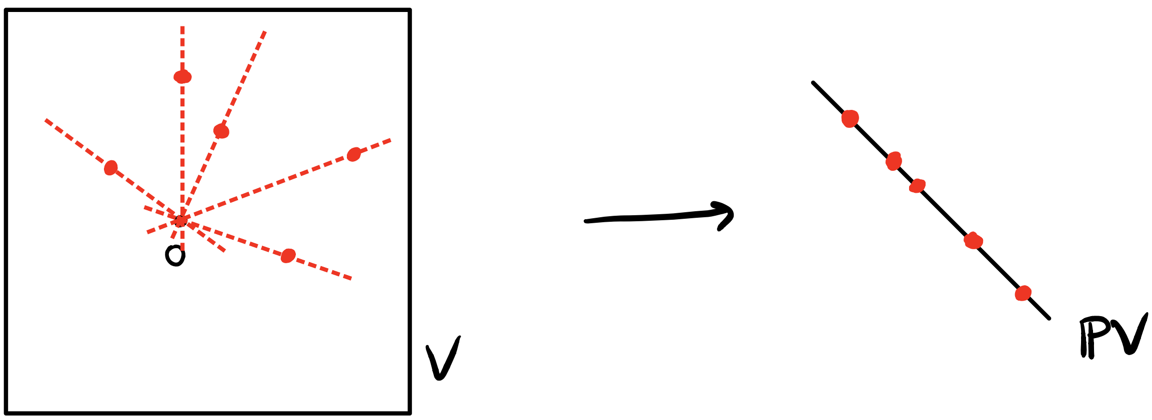
Rank 1:

$\mathcal{A}$ = one non-zero vector

$\mathbb{P}\mathcal{A} = \mathbb{P}V$ = one point

$$\Rightarrow M(\mathcal{A}) = U_{1,1}$$

Rank 2:



$\mathcal{A}$ = finite set of pairwise lin. indep. vectors in a plane

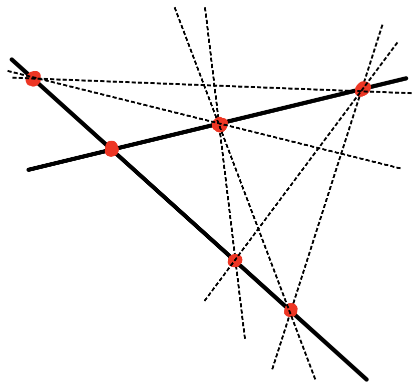
$\mathbb{P}\mathcal{A}$ = finite set of collinear points

$$\Rightarrow M(\mathcal{A}) = U_{2,n}$$

Rank 3:

$\mathcal{A}$ = finite set of vectors in 3-space s.t. each pair of vectors spans a plane $\rightsquigarrow$ $\mathbb{P}\mathcal{A}$ = finite set of points in a projective plane

Convention: In $\mathbb{P}\mathcal{A}$, only draw the lines containing ≥ 3 points

Ex: 

Many possibilities

$\mathcal{B}(M(\mathcal{A})) \longleftrightarrow$ sets of 3 non-collinear points

$\mathcal{C}(M(\mathcal{A})) \longleftrightarrow$ sets of 3 collinear points
and sets of 4 points, no 3 of which are collinear

Rank 0 flat $= \emptyset$

Rank 1 flats $\longleftrightarrow$ points $\longleftrightarrow$ ground set

Rank 2 flats $\longleftrightarrow$ lines

Rank 3 flat $= E$

Rank 4: similar – convention is to draw only "interesting" lines + planes