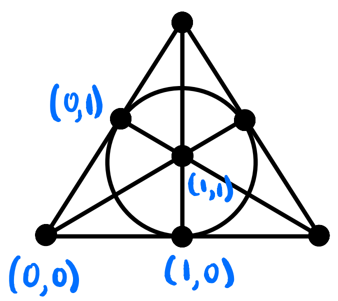
Geometric Representation

Last Week: Any finite subset of projective space defines a representable simple matroid.

Ex: The Fano matroid F_7 is the matroid of the configuration of all 7 points in $\mathbb{P}^2_{\mathbb{F}_2}$:



7 pts $\leftrightarrow$ ground set $\leftrightarrow$ rank 1 flats

7 lines $\leftrightarrow$ 3 element circuits $\leftrightarrow$ rank 2 flats

Each pair of points collinear with a third.

$\binom{7}{3} - 7 = 28$ bases are the sets of 3 non-collinear points.

More generally, we can give a <u>geometric representation</u> of any simple matroid of rank ≤ 4 as a collection of

points $\leftrightarrow$ rank 1 flats $\leftrightarrow$ elements of ground set

lines $\leftrightarrow$ rank 2 flats

planes $\leftrightarrow$ rank 3 flats

Note: lines and planes can be "twisted"

Rules for Geometric Representations

Non-degeneracy conditions

- Every line contains at least 2 points
- Any 2 distinct points lie on a line
- Every plane contains at least 3 non-collinear points
- Any 3 distinct non-collinear points lie on a plane

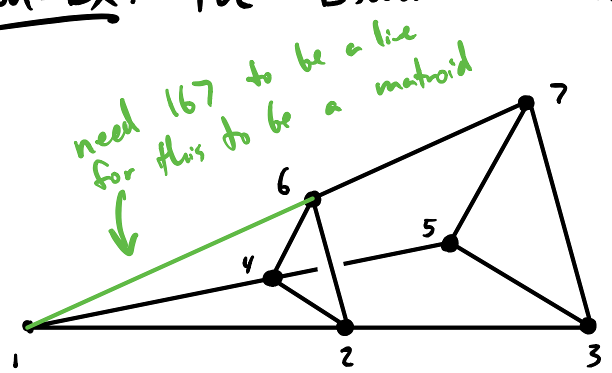
In dimension 2 (one plane)

- Any 2 distinct lines meet in at most one point

In dimension 3 (more than one plane)

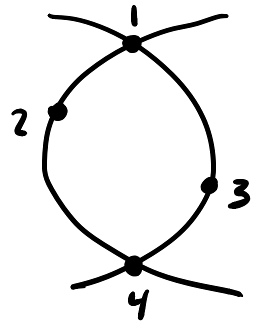
- Any 2 distinct lines meeting in a point do so in at most one point <u>and</u> lie on a common plane
- Any 2 distinct planes meeting in more than 2 points do so in a line
- Any line not lying on some plane intersects that plane in at most one point

Non-Ex: The Escher "matroid"



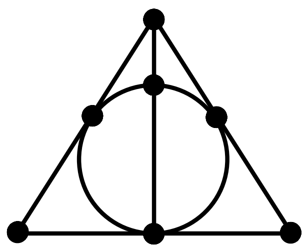
- 12367 and 14567 must be planes. But they intersect in 167, which is not a line.
- Another perspective: 167 is independent but it can't be augmented by the larger ind. set 1246

Non-Ex:



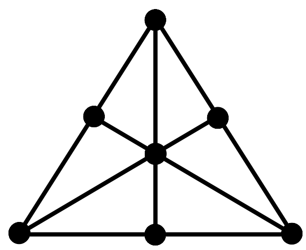
- Two distinct lines 124 and 134 meet in 2 distinct points
- 14 is independent, but it can't be augmented by 123.

Non-Ex:



Exercise

Ex: The **non-Fano matroid** F_7^- is the matroid with geometric rep.



6 3-pointed lines
3 2-pointed lines

Thm: F_7 is K-representable $\iff$ char $K = 2$

F_7^- is K-representable $\iff$ char $K \neq 2$