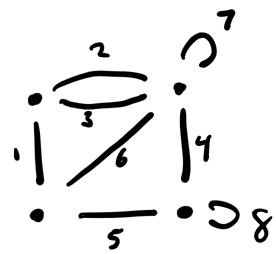
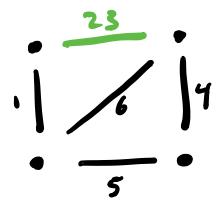
Representing non-simple matroids

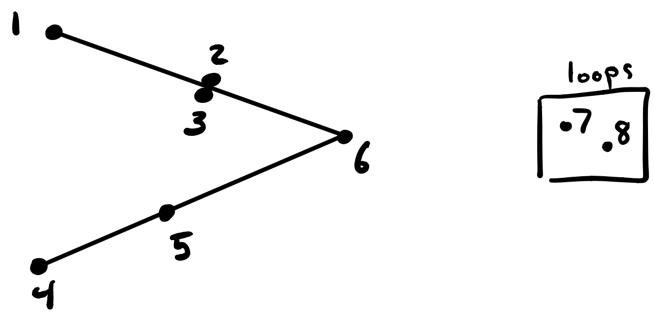
If M is not simple, modify the geometric rep. of $\tilde{M}$ by

- adding loops "in a box"
- indicate parallel elements as multi-points

Ex: $G =$ 

$\tilde{G} =$ 

Geometric rep. of $M(G)$:



Strong Basis Exchange

Recall: M matroid, $\mathcal{B}$ its collection of bases

(B2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$.

Lemma: Let M be a matroid. Then $\mathcal{B} = \mathcal{B}(M)$ satisfies

(B2)* If $B_1, B_2 \in \mathcal{B}$ and $y \in B_2 \setminus B_1$, then there exists $x \in B_1 \setminus B_2$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$.

Proof: $y \notin B_1$, so we have the fundamental circuit (Day 2 Exercise 3):

$$C(y, B_1) \subseteq B_1 \cup y.$$

Since $C(y, B_1) \not\subseteq B_2$, there exists

$$x \in C(y, B_1) \setminus B_2 \subseteq B_1 \setminus B_2$$

Now, $C(y, B_1)$ is the <u>unique</u> circuit in $B_1 \cup y$, but

$$C(y, B_1) \not\subseteq (B_1 \setminus x) \cup y \subseteq B_1 \cup y.$$

So $(B_1 \setminus x) \cup y$ is independent and $|(B_1 \setminus x) \cup y| = |B_1|$, thus $(B_1 \setminus x) \cup y \in \mathcal{B}$. $\blacksquare$

Thm: Let M be a matroid on E. Then

$$\mathcal{B}^*(M) = \{ E \setminus B \mid B \in \mathcal{B}(M) \}$$

is the set of bases of a matroid on E.

Proof: Show $\mathcal{B}^*(M)$ satisfies (B1) and (B2).

(B1): $\mathcal{B}(M) \neq \emptyset \implies \mathcal{B}^*(M) \neq \emptyset$.

(B2): Let $B_1^*, B_2^* \in \mathcal{B}^*(M)$, so that

$$B_1^* = E \setminus B_1, \quad B_2^* = E \setminus B_2$$

for some $B_1, B_2 \in \mathcal{B}(M)$.

If $x \in B_1^* \setminus B_2^* = B_2 \setminus B_1$, then by $(B2)^*$ there exists $y \in B_1 \setminus B_2 = B_2^* \setminus B_1^*$ such that

$$(B_1 \setminus y) \cup x \in \mathcal{B}(M).$$

But then

$$E \setminus ((B_1 \setminus y) \cup x) = (E \setminus (B_1 \cup x)) \cup y$$
$$= (B_1^* \setminus x) \cup y \in \mathcal{B}^*(M).$$

∎

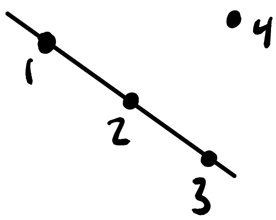
Def: Given a matroid M on E, the matroid on E with bases $\mathcal{B}^*(M)$ is called the <u>dual matroid</u> of M, denoted M^*.

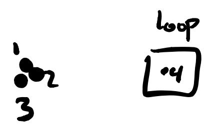
i.e. $\mathcal{B}(M^*) = \mathcal{B}^*(M)$

Ex: $(M^*)^* = M$ for any M.

Ex: $U_{r,n}^* = U_{n-r,n}$

Ex: The dual of a simple matroid need not be simple:

$M =$ 

$\rightarrow$ $M^* =$ 

$\mathcal{B}(M) = \{124, 134, 234\}$

$\mathcal{B}^*(M) = \{3, 2, 1\}$