<u>Last time</u>: If M is a matroid, then

$$\mathcal{B}^*(M) = \{E \setminus B \mid B \in \mathcal{B}(M)\}$$

is the set of bases of a matroid M^*, the <u>dual</u> of M.

$$rk(M^*) = |E| - rk(M)$$

<u>Big question</u>: What properties are preserved by duality?

<u>Ex</u>: M simple $\not\Rightarrow$ M^* is simple

- $U_{n,n}$ is simple, but $U_{n,n}^* = U_{0,n}$ consists of n loops.
- $U_{n-1,n}$ is simple (if $n \geq 3$), but $U_{n-1,n}^* = U_{1,n}$ consists of n parallel elements.

<u>Def</u>: Members of $\mathcal{B}^*(M)$ are called <u>cobases</u> of M.
Similarly,

circuits of M^* = <u>cocircuits</u> of M = $\mathcal{C}^*(M)$

independent sets of M^* = <u>coindependent sets</u> of M = $\mathcal{I}^*(M)$

hyperplanes of M^* = <u>cohyperplanes</u> of M = $\mathcal{H}^*(M)$

$\vdots$

<u>Ex</u>: e is a <u>loop</u> in M (e is in no bases) $\iff$ e is a <u>coloop</u> in M^* (e is in every basis)

Prop: Let M be a matroid on E, $X \subseteq E$.

① X is independent $\iff$ $E \setminus X$ is cospanning

② X is spanning $\iff$ $E \setminus X$ is coindependent

③ X is a hyperplane $\iff$ $E \setminus X$ is a cocircuit

④ X is a circuit $\iff$ $E \setminus X$ is a cohyperplane

Proof: ① $X \in \mathcal{I}(M) \iff \exists\, B \in \mathcal{B}(M)$ s.t. $X \subseteq B$

$\iff \exists\, B \in \mathcal{B}(M)$ s.t. $E \setminus X \supseteq E \setminus B$

$\iff \exists\, B^* \in \mathcal{B}^*(M)$ s.t. $E \setminus X \supseteq B^*$

$\iff$ $E \setminus X$ spans M^*.

② is ① applied to M^*.

③ $X \in \mathcal{H}(M) \iff X$ is non-spanning in M, but $X \cup e$ spans M for every $e \in E \setminus X$

$\overset{②}{\iff} E \setminus X \notin \mathcal{I}^*(M)$, but

$E \setminus (X \cup e) = (E \setminus X) \setminus e \in \mathcal{I}^*(M)$

for every $e \in E \setminus X$.

$\iff E \setminus X \in \mathcal{C}^*(M)$.

④ is ③ applied to M^*.

Cor: Let E be a finite set and $\mathcal{H} \subseteq 2^E$. Then $\mathcal{H} = \mathcal{H}(M)$ for some matroid M on E if and only if $\mathcal{H}$

(H1) $E \notin \mathcal{H}$.

(H2) If $H_1, H_2 \in \mathcal{H}$ with $H_1 \subseteq H_2$, then $H_1 = H_2$.

(H3) If $H_1, H_2 \in \mathcal{H}$, $H_1 \neq H_2$, and $e \in E \setminus (H_1 \cup H_2)$, then there exists $H \in \mathcal{H}$ with

$$(H_1 \cap H_2) \cup e \subseteq H.$$

Ex: (Friday Exercise #3)

If G is a graph, then

$$\mathcal{C}^*(M(G)) = \{X \subseteq E(G) \mid X \text{ is a } \underline{\text{cutset}}\}$$

minimal set of edges whose removal increases the # of conn. components

Thus, $M(G)^*$ is sometimes called the <u>cutset matroid</u> or <u>bond matroid</u> of G (as opposed to the <u>cycle matroid</u> $M(G)$).

A matroid isomorphic to $M(G)^*$ for some G is called a <u>cographic matroid</u>.

Ex: $M(K_5)^*$ is not graphic.

Suppose $M(K_5)^* \cong M(G)$. WLOG, G is connected (Friday Exercise 1)

$rk(M(K_5)) = |V(K_5)| - 1 = 4$

and $|E(K_5)| = 10$.

$\Rightarrow M(K_5)^*$ is a rank $10 - 4 = 6$ matroid on 10 elements.

So G has 10 edges and 7 vertices

$\Rightarrow G$ has a vertex of valence ≤ 2

[only have 20 edge ends to distribute among 7 vertices]

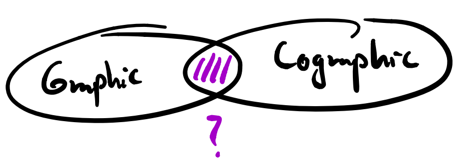
$\Rightarrow G$ has a cutset of size ≤ 2

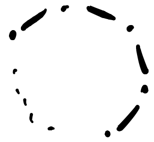
$\Rightarrow M(G)^* = M(K_5)$ has a circuit of size ≤ 2

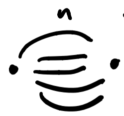
This is false, so no such G exists.

Cor: $M(K_5)$ is not cographic

Q: What is the overlap



Ex: $U_{n-1,n} = M\Big($  $\Big)$

$U_{n-1,n}^{*} = U_{1,n} = M\Big($  $\Big)$

So $U_{n-1,n}$ and $U_{1,n}$ are in the overlap.

Thm (Whitney 1932): $M(G)$ is cographic $\iff$ G is a planar graph.