Fun facts about cocircuits (Exercises)

Let M be a matroid on E, $X \subseteq E$.

circuit-cocircuit orthogonality

- $X \in \mathcal{C}^*(M) \iff X$ is a minimal non-empty set with the property that $|X \cap C| \neq 1$ for every $C \in \mathcal{C}(M)$.
- $X \in \mathcal{C}(M) \iff X$ is a minimal non-empty set with the property that $|X \cap C^*| \neq 1$ for every $C^* \in \mathcal{C}^*(M)$.

- $X \in \mathcal{C}^*(M) \iff X$ is a minimal non-empty set with the property that $X \cap B \neq \emptyset$ for every $B \in \mathcal{B}(M)$
- $X \in \mathcal{C}(M) \iff X$ is a minimal non-empty set with the property that $X \cap B^* \neq \emptyset$ for every $B^* \in \mathcal{B}^*(M)$

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The dual rank function

Let M be a matroid on E, $rk = rk_M : 2^E \to \mathbb{Z}$ its rank function, and $rk^* = rk_{M^*} : 2^E \to \mathbb{Z}$ the rank function of M^*.

Thm: For all $X \subseteq E$,

$$rk^*(X) = rk(E \setminus X) + |X| - rk(M).$$

2 more natural notions: In a matroid M,

- the corank of $X \subseteq E$ is

$$crk(X) := rk(M) - rk(X)$$

= minimum # of elts to add to X to make a spanning set

- the nullity of $X \subseteq E$ is

$$null(X) := |X| - rk(X)$$

= minimum # of elts to remove from X to make an indep. set.

Observe:

$$crk(X) = null^*(E \setminus X)$$
$$null(X) = crk^*(E \setminus X)$$

Proof: X spans $\iff$ $E \setminus X$ is coindependent

Proof of Thm: We already know

$$\mathrm{rk}(M^*) = |E| - \mathrm{rk}(M).$$

Let $X \subseteq E$. Then

$$\mathrm{crk}^*(X) = \mathrm{null}(E \setminus X)$$

$$\Rightarrow \quad \mathrm{rk}(M^*) - \mathrm{rk}^*(X) = |E \setminus X| - \mathrm{rk}(E \setminus X)$$

So

$$\begin{aligned} \mathrm{rk}^*(X) &= \mathrm{rk}(E \setminus X) - |E \setminus X| + \mathrm{rk}(M^*) \\ &= \mathrm{rk}(E \setminus X) - (\cancel{|E|} - |X|) + (\cancel{|E|} - \mathrm{rk}(M)) \\ &= \mathrm{rk}(E \setminus X) + |X| - \mathrm{rk}(M). \end{aligned}$$

$\blacksquare$

Note: • $\mathrm{rk}^*(X)$ = size of largest coindependent subset of X
= size of largest subset of X whose complement spans M.

• $\mathrm{rk}^{**}(X) = \mathrm{rk}(X)$

• Can check that rk satisfies (R1) – (R3) if and only if rk^* does.