Duals of representable matroids

Let $\mathcal{A} = \{v_e \mid e \in E\}$ be a configuration in a K-vector space V.
WLOG assume $\mathcal{A}$ spans V.

Naive duality: Dualize $V \rightsquigarrow V^*$

$\mathcal{A} \rightsquigarrow$ arrangement of hyperplanes in V^*

This is just another manifestation of $M(\mathcal{A})$. [Exercise 1]

Instead: Let $\{\delta_e \mid e \in E\}$ be the standard basis of K^E.
We have a short exact sequence

$$\begin{array}{ccccccccc} 0 & \to & W & \longrightarrow & K^E & \longrightarrow & V & \to & 0 \\ & & & & \delta_e & \longmapsto & v_e & & \end{array}$$

where $W = \ker(K^E \to V)$.
Now dualize:

$$\begin{array}{ccccccccc} 0 & \leftarrow & W^* & \longleftarrow & (K^E)^* & \longleftarrow & (V)^* & \leftarrow & 0 \\ & & g_e & \longleftarrow\!\shortmid & \delta_e^* & & & & \end{array}$$

This gives a configuration $\mathcal{A}' = \{g_e \mid e \in E\}$ in W^*, called the Gale dual of $\mathcal{A}$.

Thm: Let $X \subseteq E$. Then

$\{v_e \mid e \in X\}$ is lin. indep. in V $\iff$ $\{g_e \mid e \in E \setminus X\}$ spans W^*

Cor: $\{v_e \mid e \in X\}$ is a basis of $V \iff \{g_e \mid e \in E \setminus X\}$ is a basis of W^*.

Cor: M is K-representable $\iff$ M^* is K-representable.

Proof: $M(A)^* = M(A')$ by above corollary.

See Exercise 2.

Note: In coordinates, let E be the columns of

$$A = [v_1 | v_2 | \cdots | v_n].$$

$$0 \to W \to K^E \xrightarrow{A} V \to 0$$

- Row ops + col scaling + col swapping (change of basis)

  $A \rightsquigarrow [I_r \mid B]$ which has the same matroid as A

- Then M^* is the matroid of column vectors of

  $[I_{n-r} \mid -B^T]$.

- $\text{Rowspace}([I_r \mid B]) = \text{Rowspace}([I_{n-r} \mid -B^T])^\perp$

$$V^* = W^\perp$$

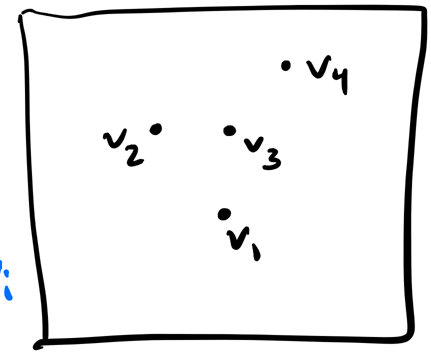
as subspaces of $K^E \cong (K^E)^*$

Ex: $A = \begin{bmatrix}1&0&1&2\\0&1&1&2\end{bmatrix}$

$V \cong K^2$

$$0 \to \underset{\ker(K^E \to V)}{\overset{\cong K^2}{W}} \xrightarrow{\begin{bmatrix}1&0\\1&0\\-1&2\\0&-1\end{bmatrix}} K^4 \xrightarrow{A} V \to 0$$

$\delta_i \longmapsto v_i$



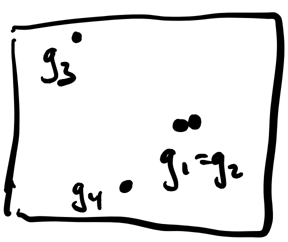
$$\begin{bmatrix}1&0\\1&0\\-1&2\\0&-1\end{bmatrix}$$

$v_1 + v_2 - v_3 = 0$

$2v_3 - v_4 = 0$

$M(A)^*$ is the matroid of the configuration

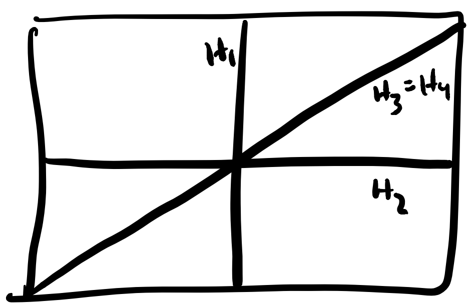
$$W^* \xleftarrow{\begin{bmatrix}1&1&-1&0\\0&0&2&-1\end{bmatrix}} \boxed{K^4 \xleftarrow{\begin{bmatrix}1&0\\0&1\\1&1\\2&2\end{bmatrix}} V^*}$$



Embeds $V^* \cong K^2$ into K^4

+ the intersection of V^* w/ coordinate hyperplanes gives a line arrangement in V^*

V^*



$H_1 = \begin{bmatrix}1\\0\end{bmatrix}^\perp$

$H_2 = \begin{bmatrix}0\\1\end{bmatrix}^\perp$

$H_3 = \begin{bmatrix}1\\1\end{bmatrix}^\perp$

$H_4 = \begin{bmatrix}2\\2\end{bmatrix}^\perp$