The lattice of flats

Wednesday Exercise 3:

Thm: Let E be a finite set and $\mathcal{F} \subseteq 2^E$. Then $\mathcal{F} = \mathcal{F}(M)$ for some matroid M on E if and only if

(F1) $E \in \mathcal{F}$

(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.

(F3) If $F \in \mathcal{F}$ and $G_1, \dots, G_k \in \mathcal{F}$ are the members of $\mathcal{F}$ which cover F, then

$$E \setminus F = E \setminus G_1 \sqcup \cdots \sqcup E \setminus G_k$$

G covers F if $F \subsetneq G$ and if $F \subseteq H \subseteq G$ for $H \in \mathcal{F}$, then $H = F$ or $H = G$.

Notation: $F \lessdot G$

We'll partially order $\mathcal{F}(M)$ by inclusion

A poset is a set P with partially defined order $\leq$ which is

- reflexive $(x \leq x \quad \forall\ x \in P)$
- antisymmetric $(x \leq y$ and $y \leq x \Rightarrow x = y)$
- transitive $(x \leq y$ and $y \leq z \Rightarrow x \leq z)$

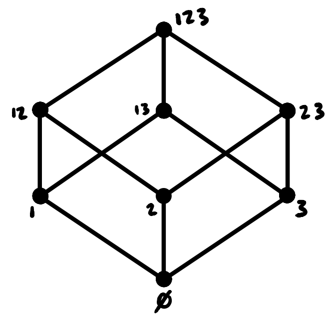
Call this poset $\mathcal{L}(M)$.

Visualize $\mathcal{L}(M)$ (or any poset) by drawing its <u>Hasse diagram</u>:

- vertices = flats
- draw an edge from F up to flat G if $F \lessdot G$

<u>Ex</u>: The poset of subsets of $\{1, 2, 3\}$

$= \mathcal{L}(U_{3,3})$

$=$ 

without the labels

(but we usually draw the labels)

A <u>lattice</u> is a poset $\mathcal{L}$ such that for all $x, y \in \mathcal{L}$ there exists $x \vee y \in \mathcal{L}$ satisfying

- $x \vee y \geq x$ and $x \vee y \geq y$
- if $z \geq x$ and $z \geq y$, then $z \geq x \vee y$

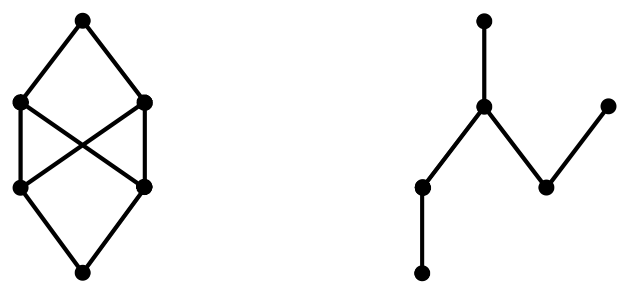
and there exists $x \wedge y \in \mathcal{L}$ satisfying

- $x \wedge y \leq x$ and $x \wedge y \leq y$
- if $z \leq x$ and $z \leq y$, then $z \leq x \wedge y$.

$x \vee y$ is the <u>join</u>, or <u>least upper bound</u>, of x and y.

$x \wedge y$ is the <u>meet</u>, or <u>greatest lower bound</u>, " " .

Ex: Not all posets are lattices



Lemma: $\mathcal{L}(M)$ is a lattice, via

$$F \vee G = \mathrm{cl}(F \cup G) \quad \text{and} \quad F \wedge G = F \cap G$$

Proof: The only nontrivial part is showing $F \cap G$ is a flat, which is (F2). ∎

Def: Let P be a poset. An element $z \in P$ is a zero if $z \leq x$ for all $x \in P$. If it exists, it is unique and denoted 0_P.
Similarly, a one is the (unique if it exists) element $1_P \in P$ such that $x \leq 1_P$ for all $x \in P$.

Ex: In a finite lattice $\mathcal{L}$,
$$0_{\mathcal{L}} = \bigwedge_{x \in \mathcal{L}} x$$
$$1_{\mathcal{L}} = \bigvee_{x \in \mathcal{L}} x.$$

Ex: In $\mathcal{L}(M)$, the zero is

$$cl(\emptyset) = \{\text{loops}\}.$$

and the one is the ground set E.

In a poset, a <u>chain</u> from x to y is a sequence

$$x = x_0 < x_1 < x_2 < \cdots < x_k = y.$$

Its <u>length</u> is k, and it is <u>maximal</u> if

$x_{i-1} \lessdot x_i$ for all i.

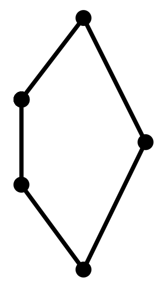
In a poset with 0, the <u>height</u> $h(x)$ of x is the length of the longest chain from 0 to x. An element is an <u>atom</u> if it has height 1 (i.e. an atom covers 0).

<u>Lemma</u>: Let M be a matroid.

1. Every flat F is a join of atoms.
2. If $F \subseteq G$ are flats, then every maximal chain from F to G has length $rk(G) - rk(F)$.

Proof: Exercise 2, Wednesday.

Ex: The following is not $\mathcal{L}(M)$ for any M:



Cor: The height function on $\mathcal{L}(M)$ is rk_M.

Def: A lattice $\mathcal{L}$ is *geometric* if

- $\mathcal{L}$ is finite
- $\mathcal{L}$ is *atomic*: each $x \in \mathcal{L}$ is the join of atoms
- the height function of $\mathcal{L}$ is *submodular*:

$$h(x \vee y) + h(x \wedge y) \leq h(x) + h(y).$$

- $\mathcal{L}$ has the *Jordan-Dedekind property*: if $x < y$ in $\mathcal{L}$, then every maximal chain from x to y has the same length.

Thm: A lattice is geometric if and only if it is the lattice of flats of a matroid.

Proof (sketch): ($\Leftarrow$) ✓

($\Rightarrow$) Trivial case: $\mathcal{L} = \bullet = \mathcal{L}(U_{0,0})$

Else, set $E = \{\text{atoms of } \mathcal{L}\}$ and show

$$\begin{aligned} rk: 2^E &\longrightarrow \mathbb{Z} \\ X &\longmapsto h\left(\bigvee_{x \in X} x\right) \end{aligned}$$

satisfies (R1) - (R3), so it is the rank function of a matroid M on E.

Last step: $\mathcal{L}(M) \cong \mathcal{L}$. ∎

Cor: $\mathcal{L}(M_1) \cong \mathcal{L}(M_2) \Leftrightarrow \widetilde{M}_1 \cong \widetilde{M}_2$.