Last time:

Def: A lattice $\mathcal{L}$ is <u>geometric</u> if

- $\mathcal{L}$ is finite
- $\mathcal{L}$ is <u>atomic</u>: each $x \in \mathcal{L}$ is a join of atoms
- the height function of $\mathcal{L}$ is <u>submodular</u>:

$$h(x \vee y) + h(x \wedge y) \leq h(x) + h(y)$$

- $\mathcal{L}$ has the <u>Jordan-Dedekind property</u>: if $x < y$ in $\mathcal{L}$, then every maximal chain from x to y has the same length.

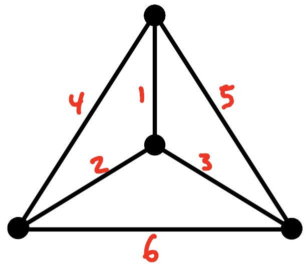
Thm: A lattice is geometric if and only if it is the lattice of flats of a matroid.

We now have 3 (unfortunately similar) ways to "draw" a matroid M:

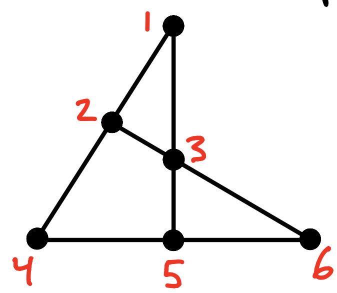
- as a graph (if M is graphic)
- geometric representation (if $\mathrm{rk}\, M \leq 4$)
- the lattice of flats

Ex: $M = M(K_4)$

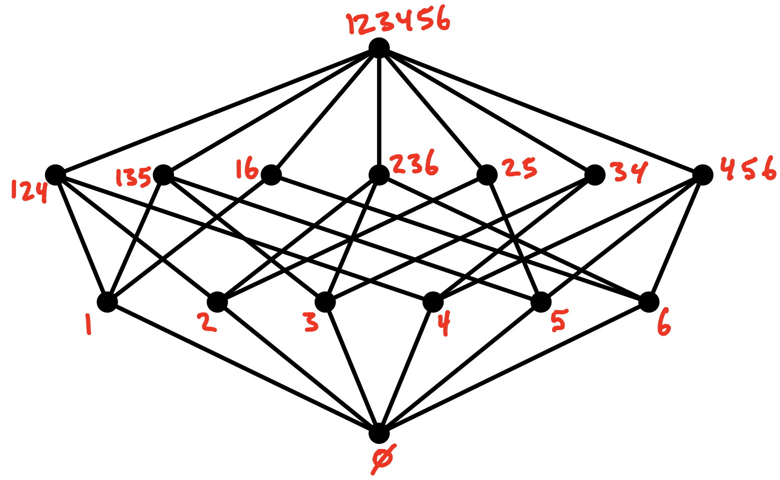
Graph



Geometric Rep.

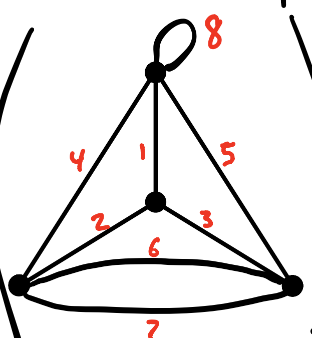


$\mathcal{L}(M)$

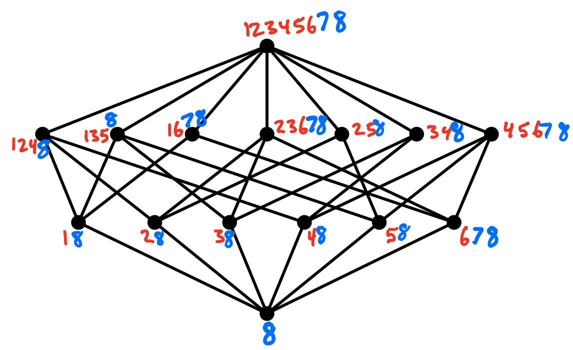


What if we add loops or parallel elements?

$M = M\Big($



$\Big) \to \mathcal{L}(M) =$



Only the labels change.

Last time: $\mathcal{L}(M_1) \cong \mathcal{L}(M_2) \iff \widetilde{M}_1 \cong \widetilde{M}_2$

Thm: Let M be a matroid and $F \in \mathcal{F}(M)$. If

$$\mathrm{crk}(F) = \mathrm{rk}(M) - \mathrm{rk}(F) = k$$

then there exist hyperplanes $H_1, \ldots, H_k \in \mathcal{H}(M)$ such that

$$F = \bigcap_{i=1}^{k} H_i .$$

That is, $\mathcal{L}(M)$ is coatomic.

Proof: Induct on k. $k=0$ is trivial (E = empty intersection)

If $k \geq 1$, choose $e \in E \setminus F$. Then $\mathrm{cl}(F \cup e)$ is a flat covering F. Thus,

$$\begin{aligned} \mathrm{rk}(\mathrm{cl}(F \cup e)) &= \mathrm{rk}(F) + 1 = (\mathrm{rk}(M) - k) + 1 \\ &= \mathrm{rk}(M) - (k-1) \end{aligned}$$

i.e. $\mathrm{cl}(F \cup e)$ has corank $k-1$.

Let $H_1, \ldots, H_{k-1} \in \mathcal{H}(M)$ be hyperplanes with

$$\mathrm{cl}(F \cup e) = \bigcap_{i=1}^{k-1} H_i .$$

Now, there exists $H_k \in \mathcal{H}(M)$ with

$$F \subseteq H_k \subseteq E \setminus e .$$

Why? $F \subseteq E \setminus e$ is non-spanning, so it is contained in a maximal non-spanning subset of $E \setminus e$.

Now,

$$cl(F \cup e) = \bigcap_{i=1}^{k-1} \underset{e}{\overset{}{H_i}} \supsetneq \bigcap_{i=1}^{k} H_i \supseteq F.$$

But these are all flats, and $cl(F \cup e)$ covers F.

So $F = \bigcap_{i=1}^{k} H_i$. ∎

Flats and duality

Cor: F is a flat of a matroid M if and only if $E \setminus F$ is a union of cocircuits.

Proof: $F \in \mathcal{F}(M) \iff F = \bigcap_{i=1}^{k} H_i$, each $H_i \in \mathcal{H}(M)$

$\iff E \setminus F = \bigcup_{i=1}^{k} (E \setminus H_i)$, each $E \setminus H_i \in \mathcal{C}^*(M)$. ∎