Matroid Minors

Def/Thm: Let M be a matroid on E, and $X \subseteq E$.

Then
$$\{ I \subseteq X \mid I \in \mathcal{I}(M) \}$$
is the collection of independent sets of a matroid on X.

This, denoted $M|X$, is the <u>restriction of M to X</u>.

Proof: Easy exercise.

Observations: • $\mathcal{B}(M|X)$ = maximal independent subsets of X
= maximal members of $\{ B \cap X \mid B \in \mathcal{B}(M) \}$

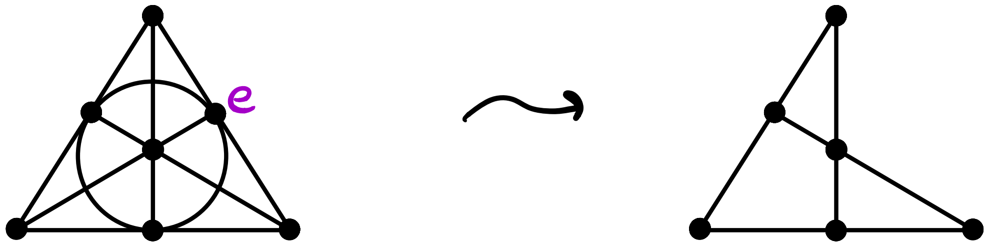
• $\mathcal{C}(M|X) = \{ C \subseteq X \mid C \in \mathcal{C}(M) \}$

• $rk_{M|X} = rk_M\big|_{2^X}$. That is, if $S \subseteq X$, then
$$rk_{M|X}(S) = rk_M(S).$$

In particular, $rk(M|X) = rk_M(X)$.

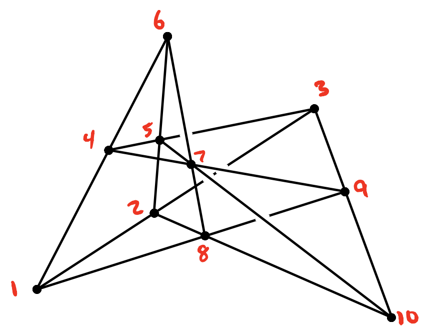
Note: Let $T = E \setminus X$. Sometimes $M|X$ is called the <u>deletion</u> of T and denoted $M \setminus T$.

Ex (Lecture 8 Exercise 3): Any single-element deletion of F_7 is $M(K_4)$.

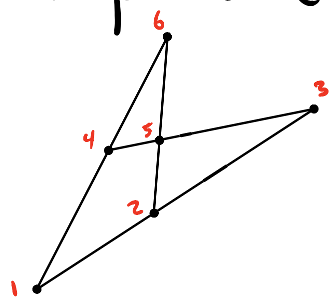


F_7 $\qquad\qquad$ $F_7 \setminus e \cong M(K_4)$

Ex: The restriction of the Desargues matroid $(= M(K_5))$

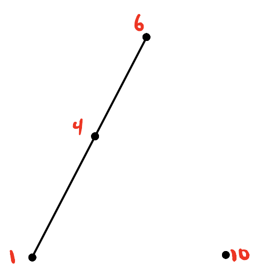


to a 6-element plane (like 1,2,3,4,5,6) is $M(K_4)$.



($\leftrightarrow$ K_4 subgraphs)

The restriction to a 4-element plane (like 1,4,6,10) looks like



($\leftrightarrow$ $K_2 \sqcup K_3$ subgraphs)

A dual construction:

Def: Let M be a matroid on E, and let $T \subseteq E$. The <u>contraction of M by T</u> is the matroid

$$M/T := (M^* \setminus T)^*$$

on ground set $E \setminus T$.

The rank of M/T satisfies

$$rk(M/T) + rk(M^* \setminus T) = |E \setminus T|$$

$$\begin{aligned} \Rightarrow rk(M/T) &= |E \setminus T| - rk(M^* \setminus T) \\ &= |E \setminus T| - rk_{M^*}(E \setminus T) \\ &= |E \setminus T| - (rk_M(T) + |E \setminus T| - rk(M)) \\ &= rk(M) - rk_M(T) \\ &= crk_M(T). \end{aligned}$$

Thm: For $S \subseteq E \setminus T$,

$$rk_{M/T}(S) = rk_M(S \cup T) - rk_M(T)$$

Proof: Exercise.

It follows that

- $\mathcal{I}(M/T) = \{ I \subseteq E \setminus T \mid B \cup I \in \mathcal{I}(M) \text{ for some } B \in \mathcal{B}(M|T) \}$
- $\mathcal{B}(M/T) = \{ B' \subseteq E \setminus T \mid B \cup B' \in \mathcal{B}(M) \text{ for some } B \in \mathcal{B}(M|T) \}$
- $\mathcal{C}(M/T) =$ minimal non-empty members of $\{ C \setminus T \mid C \in \mathcal{C}(M) \}$