Last time:

M matroid on E, $T \subseteq E$.

Deletion: $M \setminus T$, matroid on $E \setminus T$ with

$$rk_{M\setminus T}(S) = rk_M(S) \quad \text{for all } S \subseteq E \setminus T$$

[Equivalently, if $X = E \setminus T$, then $M \setminus T = M|X$ is the restriction of M to X.]

Contraction: $M/T = (M^* \setminus T)^*$, matroid on $E \setminus T$ with

$$rk_{M/T}(S) = rk_M(S \cup T) - rk_M(T) \quad \text{for all } S \subseteq E \setminus T.$$

Basic Properties

If $T_1, T_2 \subseteq E$ are disjoint, then

- $(M \setminus T_1) \setminus T_2 = M \setminus (T_1 \cup T_2) = (M \setminus T_2) \setminus T_1$
- $(M / T_1) / T_2 = M / (T_1 \cup T_2) = (M / T_2) / T_1$
- $(M / T_1) \setminus T_2 = (M \setminus T_2) / T_1$

$\Rightarrow$ any sequence of deletions and contractions can be written as $M / T_1 \setminus T_2$

Def: A matroid of the form $M / T_1 \setminus T_2$ $(T_1 \cap T_2 = \emptyset)$ is called a **minor** of M.

Ex: Let $T \subseteq [n]$ with $|T| = k$. Then

$$U_{r,n} \setminus T = \begin{cases} U_{r,n-k} & \text{if } k < n-r \quad (|E \setminus T| > r) \\ U_{n-k,n-k} & \text{if } n-r \le k \le n \quad (E \setminus T \text{ is indep.}) \end{cases}$$

and

$$U_{r,n} / T = \begin{cases} U_{r-k,n-k} & \text{if } k < r \\ U_{0,n-k} & \text{if } r \le k \le n \end{cases}$$

That is, uniform matroids form a <u>minor-closed class</u>.

Other minor-closed classes:
- graphic matroids
- cographic matroids
- K-representable matroids for a field K
- representable matroids
- regular matroids

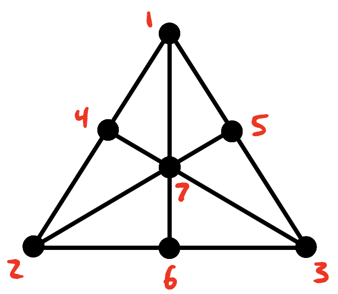
An <u>excluded minor</u> of a minor-closed class $\mathcal{M}$ is a matroid not in $\mathcal{M}$ but with all proper minors in $\mathcal{M}$.

Clearly, $\mathcal{M}$ is determined by its list of excluded minors.

<u>Hard problem</u>: Identify all excluded minors of $\mathcal{M}$.

$\mathcal{M}$	Excluded Minors	Proof
Graphic	$U_{2,4}$, F_7, F_7^*, $M(K_5)^*$, $M(K_{3,3})^*$	Tutte 1959
Cographic	$U_{2,4}$, F_7, F_7^*, $M(K_5)$, $M(K_{3,3})$	Tutte 1959
Regular	$U_{2,4}$, F_7, F_7^*	Tutte 1958
$\mathbb{F}_2$-rep'ble	$U_{2,4}$	Tutte 1958
$\mathbb{F}_3$-rep'ble	$U_{2,5}$, $U_{3,5}$, F_7, F_7^*	Bixby/Seymour (independently) 1979
$\mathbb{F}_4$-rep'ble	$U_{2,6}$, $U_{4,6}$, F_7^-, $(F_7^-)^*$, 3 others	Geelen-Gerards-Kapoor 2000
$\mathbb{F}_q$-rep'ble $q \geq 5$	Conjecturally finitely many (Rota)	Geelen-Gerards-Whittle announced 2013
K-rep'ble char $K = 0$	Infinitely many	Lazarson 1958

<u>Ex</u>: F_7^- is not $\mathbb{F}_2$-representable, so Tutte says it has a $U_{2,4}$ minor.



F_7^-

$U_{2,4}$

First: How do we draw a geometric rep. of M/e given a geometric rep. of M?

- If e is a loop or coloop: $M/e = M \backslash e$ (Exercise)

- Otherwise: Choose a hyperplane $H \not\ni e$.

  Draw M/e on H by projecting from e.

  For $f \in E \backslash e$,

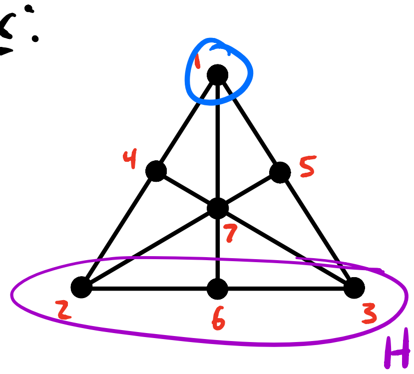
  ① If f is parallel to e in M, then f is a loop in M/e.

  $$rk_{M/e}(f) = rk_M(\{f,e\}) - rk_M(e) = 0$$

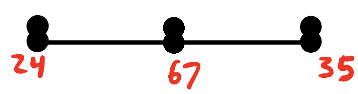
  ② If $f \in H$, then leave it where it is.

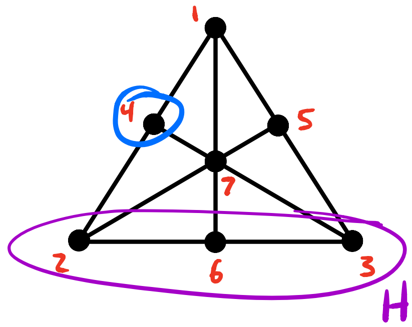
  ③ If $f \notin H \cup e$ and f is not parallel to e, project it onto the intersection point of H with $cl\{e,f\}$. Add the point if necessary.

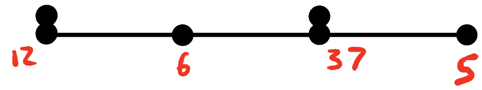
Ex:



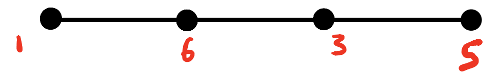
$\longrightarrow$ $F_7^-/1$ =





$\longrightarrow F_7^-/4 =$ 

$\longrightarrow F_7^-/4\backslash\{2,7\}$

$=$ 

$\cong U_{2,4}$