Direct Sums

Def/Thm: Let M_1 and M_2 be matroids on disjoint ground sets E_1 and E_2, resp. The direct sum $M_1 \oplus M_2$ is the matroid on $E_1 \sqcup E_2$ with

$$\mathcal{I}(M_1 \oplus M_2) = \{ I_1 \cup I_2 \mid I_1 \in \mathcal{I}(M_1),\ I_2 \in \mathcal{I}(M_2) \}$$

disjoint

Proof: Easy.

It's straightforward to check

- $\mathcal{B}(M_1 \oplus M_2) = \{ B_1 \cup B_2 \mid B_1 \in \mathcal{I}(M_1),\ B_2 \in \mathcal{I}(M_2) \}$
- $\mathcal{C}(M_1 \oplus M_2) = \mathcal{C}(M_1) \cup \mathcal{C}(M_2)$
- For $X_1 \subseteq E_1$ and $X_2 \subseteq E_2$,

$$rk_{M_1 \oplus M_2}(X_1 \cup X_2) = rk_{M_1}(X_1) + rk_{M_2}(X_2)$$

- $\mathcal{F}(M_1 \oplus M_2) = \{ F_1 \cup F_2 \mid F_1 \in \mathcal{F}(M_1),\ F_2 \in \mathcal{F}(M_2) \}.$
- $\mathcal{L}(M_1 \oplus M_2) \cong \mathcal{L}(M_1) \times \mathcal{L}(M_2)$

product of posets

Ex: If G_1 and G_2 are graphs, then

$$M(G_1) \oplus M(G_2) = M(G_1 \sqcup G_2) = M(G_1 \vee G_2)$$

fuse at a vertex

Ex:

G_1 $\qquad$ G_2 $\qquad$ $G_1 \dot{\vee} G_2$

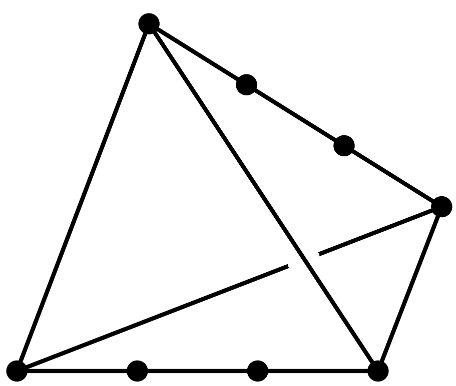
Ex: If $\mathcal{A}_1$ is a configuration in a K-vector space V_1

" $\mathcal{A}_2$ " " " " " V_2

then we get a configuration $\mathcal{A}_1 \oplus \mathcal{A}_2$ in $V_1 \oplus V_2$ via $V_i \hookrightarrow V_1 \oplus V_2$ and

$$M(\mathcal{A}_1) \oplus M(\mathcal{A}_2) = M(\mathcal{A}_1 \oplus \mathcal{A}_2)$$

Ex: $U_{2,4} \oplus U_{2,4}$ has geometric representation



This is <u>not</u> $U_{4,8}$, so we see that uniform matroids are not closed under taking direct sums.

Ex: $M \oplus U_{0,1}$ is obtained from M by adding a loop.

Ex: $M \oplus U_{1,1}$ " " " " " coloop.

Ex: $U_{0,n} = U_{0,1} \oplus \cdots \oplus U_{0,1}$

$U_{n,n} = U_{1,1} \oplus \cdots \oplus U_{1,1}$

Truncation

Def/Thm: Let M be a matroid on E of rank $\mathrm{rk}(M) \geq 1$. The truncation of M is the matroid $\mathrm{trunc}(M)$ on E with

$$\mathcal{I}(\mathrm{trunc}(M)) = \{ I \in \mathcal{I}(M) \mid |I| \leq \mathrm{rk}(M) - 1 \}$$

Proof: Easy.

Observe that if $X \subseteq E$,

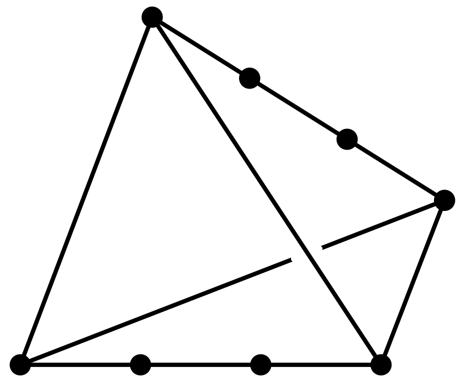
$$\mathrm{rk}_{\mathrm{trunc}(M)}(X) = \min \{ \mathrm{rk}_M(X), \mathrm{rk}(M) - 1 \}$$

Ex: $\mathrm{trunc}(U_{r,n}) = U_{r-1,n}$ if $r \geq 1$.

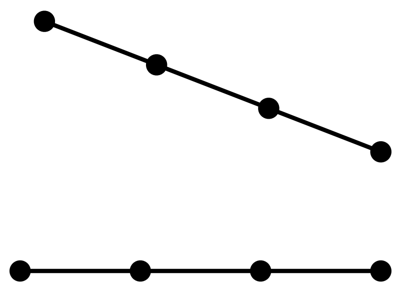
Geometrically, truncation is projection onto a generic hyperplane.

Ex:

$U_{2,4} \oplus U_{2,4}$



$\mathrm{trunc}(U_{2,4} \oplus U_{2,4})$



$\mathrm{trunc}^2(U_{2,4} \oplus U_{2,4})$

$= U_{2,8}$