Connectivity

How can we tell when a matroid has a nontrivial direct sum decomposition?

Lecture 4, Exercise 3:

M a matroid on E.

(a) $C \in \mathcal{C}(M) \iff C \subseteq E$ is a minimal non-empty set satisfying $e \in \mathrm{cl}(C \setminus e)$ for every $e \in C$.

(b) If $X \subseteq E$, then $\mathrm{cl}(X) = X \cup \{e \in E \mid \exists\, C \in \mathcal{C} \text{ with } e \in C \subseteq X \cup e\}$

An application:

Thm: Let M be a matroid and $\mathcal{C} = \mathcal{C}(M)$ its set of circuits. Then $\mathcal{C}$ satisfies

(C3)$'$ If $C_1, C_2 \in \mathcal{C}$ with $e \in C_1 \cap C_2$ and $f \in C_1 \setminus C_2$ then there is $C \in \mathcal{C}$ with

[strong circuit elimination]

$$f \in C \subseteq (C_1 \cup C_2) \setminus e.$$

Cor: $\mathcal{C} \subseteq 2^E$ satisfies the circuit axioms (C1) – (C3) if and only if it satisfies (C1), (C2), and (C3)$'$.

Proof: By (a), $e \in cl(C_2 \setminus e)$. So

$$C_2 \setminus e \subseteq (C_1 \cup C_2) \setminus \{e,f\}$$

implies $e \in cl((C_1 \cup C_2) \setminus \{e,f\})$ by (CL2).

Thus,

$$cl((C_1 \cup C_2) \setminus \{e,f\}) = cl((C_1 \cup C_2) \setminus f)$$

why? $e \in cl(X) \Rightarrow cl(X) = cl(X \cup e)$ by (CL2)+(CL3).

Now,

$$f \underset{\text{(a)}}{\in} cl(C_1 \setminus f) \underset{\text{(CL2)}}{\subseteq} cl((C_1 \cup C_2) \setminus f) = cl((C_1 \cup C_2) \setminus \{e,f\}).$$

So by (b), there is a circuit $C \in \mathcal{C}$ such that

$$f \in C \subseteq (C_1 \cup C_2) \setminus e.$$

∎

Let M be a matroid on E.

Define an equivalence relation $\sim$ on E by

$$e \sim f \iff e = f \text{ or } \{e,f\} \subseteq C \text{ for some } C \in \mathcal{C}(M)$$

Proof of transitivity:

Suppose e, f, g are distinct with $e \sim f$ and $f \sim g$.

There exist $C_1, C_2 \in \mathcal{C}(M)$ with

$$C_1 \supseteq \{e, f\}$$
$$C_2 \supseteq \{f, g\}.$$

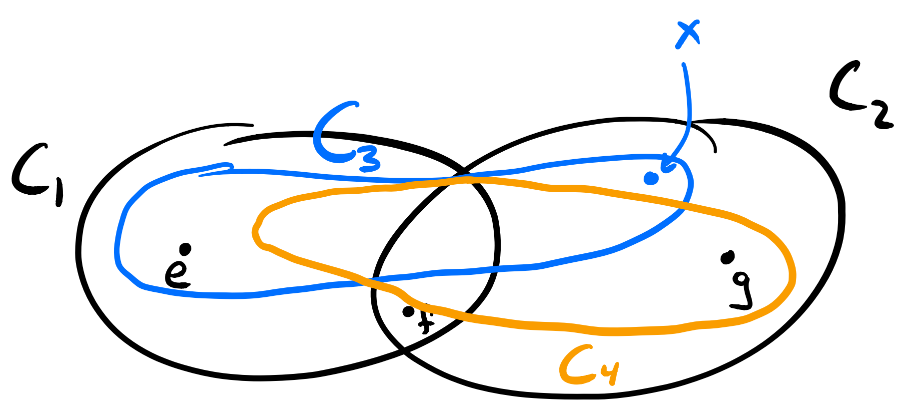
If $g \in C_1$ or $e \in C_2$, then we're done.

Otherwise, $C_1 \neq C_2$. Strong circuit elimination gives a circuit C_3 with

$$e \in C_3 \subseteq (C_1 \cup C_2) \setminus f.$$

If $g \in C_3$, we're done.

Otherwise, since $C_3 \nsubseteq C_1$, there is some $x \in (C_2 \cap C_3) \setminus C_1$.



By strong circuit elimination, there is a circuit C_4 with

$$g \in C_4 \subseteq (C_2 \cup C_3) \setminus x$$

If $e \in C_4$, we're done.

Otherwise,

$$\bullet\ C_1 \cap C_4 \supseteq (C_3 \setminus C_2) \cap C_4 \neq \emptyset$$

else $C_4 \subsetneq C_2$

$$\bullet\ C_1 \cup C_4 \subseteq (C_1 \cup C_2) \setminus x$$

$$\Rightarrow |C_1 \cup C_4| < |C_1 \cup C_2|$$

Now, repeat this process starting with C_1 and C_4. At some point, we produce a circuit C with $e, g \in C$, because $|C_1 \cup C_{2k}|$ can't decrease to 0. So $e \sim g$. ∎

Def: The $\sim$-equivalence classes are called <u>connected components</u> of M. M is <u>connected</u> if it has only one component. Otherwise, it is <u>disconnected</u>.

Ex: Every loop is a connected component.
So is every coloop.

Ex: $U_{r,n}$ is connected if $0 < r < n$

Circuits are the $(r+1)$-element subsets.