<u>Last time</u>: Equivalence relation $\sim$ on ground set of matroid M:

$$e \sim f \iff e = f \quad \text{or} \quad \{e,f\} \subseteq C \text{ for some } C \in \mathcal{C}(M).$$

Equivalence classes = <u>connected components</u> of M

M is <u>connected</u> if it has only one equivalence class. Otherwise, M is <u>disconnected</u>.

<u>Ex</u>: $M(G)$ is connected $\iff$ G is loopless, connected and has no <u>cut vertex</u>
$(|V(G)| \geq 2)$

↳ v s.t. removing v from G disconnects G

$\iff$ G is loopless and <u>2-connected</u>

≥ 2 vertices must be removed to disconnect G

<u>Ex</u>: $M(K_n)$ is connected for $n \geq 2$

Thm: Let M be a matroid on E, and $T \subseteq E$.
TFAE:

1. $M \setminus T = M / T$
2. $\mathrm{rk}(M \setminus T) \leq \mathrm{rk}(M/T)$
3. $\mathrm{rk}(T) + \mathrm{rk}(E \setminus T) = \mathrm{rk}(M)$
4. $\mathrm{rk}(T) + \mathrm{rk}^*(T) = |T|$
5. If $C \in \mathcal{C}(M)$, then either $C \subseteq T$ or $C \subseteq E \setminus T$.
6. T is a union of connected components of M.
7. $M = M|T \oplus M \setminus T$

A set T satisfying 1 - 7 is called a *separator* of M.

Cor: M is connected $\Leftrightarrow$ the only separators are $\emptyset$ and E.

Proof of Thm: Exercise.

Cor: If $T_1, \ldots, T_n$ are the connected components of M, then

$$M = M|T_1 \oplus M|T_2 \oplus \cdots \oplus M|T_n$$

and, up to isomorphism + reordering, this is the unique way to write M as a direct sum of non-empty connected matroids.

2 useful properties

① Cor: M is connected $\Leftrightarrow$ M^* is connected

Proof: ④ is self-dual.

Proof 2: $(N_1 \oplus N_2)^* = N_1^* \oplus N_2^*$

② Thm: If M is connected matroid and $e \in E$, then either $M \setminus e$ or M/e is connected.

Proof: Suppose $M \setminus e$ is not connected.

Let $x \neq y$ be arbitrary elements of $E \setminus e$.

We'll show $x \sim y$ in M/e.

Case 1: $x \not\sim y$ in $M \setminus e$

Since $x \sim y$ in M, there is a circuit $C_{xy} \in \mathcal{C}(M)$ with $x, y \in C_{xy}$.

Necessarily,

$$C_{xy} \notin \mathcal{C}(M \setminus e) = \{ C \subseteq E \setminus e \mid C \in \mathcal{C}(M) \}$$

So $e \in C_{xy}$.

We need to show

$C_{xy} \setminus e \in \mathcal{C}(M/e)$ = minimal non-empty members of $\{ C \setminus e \mid C \in \mathcal{C}(M) \}$.

Indeed,

$$\begin{aligned} C' \setminus e \subseteq C_{xy} \setminus e &\Rightarrow C' \subseteq C_{xy} \\ &\Rightarrow C' = C_{xy}. \end{aligned}$$

So $x \sim y$ in M/e.

<u>Case 2: $x \sim y$ in $M \setminus e$</u>

Since $M \setminus e$ is disconnected, there is $z \in E$ such that $x \nsim z$, $y \nsim z$ in $M \setminus e$.

Then by Case 1, $x \sim z \sim y$ in M/e. $\blacksquare$

Higher connectivity

By ④, M is connected if and only if

$$\mathrm{rk}(T) + \mathrm{rk}^*(T) \geq |T| + 1$$

for all $T \subseteq E$ such that

$$\min\{|T|, |E \setminus T|\} \geq 1. \quad (T \neq \emptyset, E)$$

Def: M is n-connected if for every $1 \leq k \leq n-1$,

$$\min\{|T|, |E\setminus T|\} \geq k$$

$$\Rightarrow \mathrm{rk}(T) + \mathrm{rk}^*(T) \geq |T| + k.$$

So ordinary connectedness = 2-connectedness.

Thm (Tutte?): Let G have at least 4 edges. Then

$M(G)$ is 3-connected $\Leftrightarrow$ G is 3-connected and simple