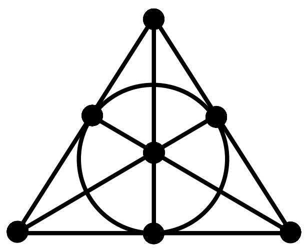
No class Monday

Last time: The characteristic polynomial of a matroid M on E is

$$\chi_M(q) = \sum_{S \subseteq E} (-1)^{|S|} q^{\operatorname{crk}(S)} \in \mathbb{Z}[q].$$

Ex: F_7



$|S| \leq 2 \Rightarrow \operatorname{rk}(S) = |S| \Rightarrow \operatorname{crk}(S) = 3 - |S|$

$|S| = 3 \Rightarrow$ either
- S is a line $\Rightarrow \operatorname{crk}(S) = 1$ $\quad \times 7$
- S is a basis $\Rightarrow \operatorname{crk}(S) = 0$ $\quad \times 28$

$|S| \geq 4 \Rightarrow \operatorname{rk}(S) = 3 \Rightarrow \operatorname{crk}(S) = 0$

So

$$\chi_{F_7}(q) = \underset{|S|=0}{q^3} - \underset{|S|=1}{7q^2} + \underset{|S|=2}{21q} - \underset{|S|=3}{(7q + 28)}$$

$$+ \underbrace{35 - 21 + 7 - 1}_{|S| \geq 4}$$

$$= q^3 - 7q^2 + 14q - 8$$

Ex: Compare $\chi_{F_7^-}(q) = q^3 - 7q^2 + 15q - 9.$

Properties

① If M has a loop, then $\chi_M(q) = 0.$

Proof: Let e be the loop. Then for any $S \subseteq E \setminus e$,

$$rk(S \cup e) = rk(S) \Rightarrow crk(S \cup e) = crk(S).$$

So

$$\chi_M(q) = \sum_{S \subseteq E} (-1)^{|S|} q^{crk(S)}$$

$$= \sum_{S \subseteq E \setminus e} (-1)^{|S|} q^{crk(S)} + \sum_{S \subseteq E \setminus e} (-1)^{|S \cup e|} q^{crk(S \cup e)}$$

$$= \left(\sum_{S \subseteq E \setminus e} (-1)^{|S|} q^{crk(S)} \right) (1 - 1)$$

$$= 0.$$

$\blacktriangle$

② If e and f are parallel in M, then

$\chi_M(q) = \chi_{M \setminus e}(q) = \chi_{M \setminus f}(q)$ [Wednesday Exercise 1]

$\Rightarrow$ If M loopless, then $\chi_M(q) = \chi_{\tilde{M}}(q).$

③ If $M \neq U_{0,0}$, then $\chi_M(1) = 0$.

↳ $\chi_{U_{0,0}}(q) = 1$

Proof: Let $e \in E$. Then

$$\chi_M(1) = \sum_{S \subseteq E} (-1)^{|S|} = \sum_{S \subseteq E \setminus e} (-1)^{|S|} + \sum_{S \subseteq E \setminus e} (-1)^{|S \cup e|}$$

$$= 0.$$

Def: The reduced characteristic polynomial of a non-empty matroid M is

$$\overline{\chi}_M(q) = \frac{\chi_M(q)}{q-1} \in \mathbb{Z}[q].$$

Ex: $$\overline{\chi}_{U_{3,4}}(q) = \frac{q^3 - 4q^2 + 6q - 3}{q-1} = q^2 - 3q + 3$$

$$\overline{\chi}_{F_7}(q) = \frac{q^3 - 7q^2 + 14q - 8}{q-1} = q^2 - 6q + 8$$

$$\overline{\chi}_{F_7^-}(q) = \frac{q^3 - 7q^2 + 15q - 9}{q-1} = q^2 - 6q + 9$$

④ If M_1, M_2 are matroids, then

$$\chi_{M_1 \oplus M_2}(q) = \chi_{M_1}(q)\, \chi_{M_2}(q).$$

Proof: $$\chi_{M_1 \oplus M_2}(q) = \sum_{S \subseteq E_1 \sqcup E_2} (-1)^{|S|} q^{\mathrm{crk}_{M_1 \oplus M_2}(S)}$$

$$= \sum_{\substack{S_1 \subseteq E_1 \\ S_2 \subseteq E_2}} (-1)^{|S_1| + |S_2|} q^{\mathrm{crk}_{M_1}(S_1) + \mathrm{crk}_{M_2}(S_2)}$$

$$= \chi_{M_1}(q)\, \chi_{M_2}(q).$$

∎

⑤ Deletion/contraction formula

Thm: Let M be a loopless matroid on E, and let $e \in E$.

If e is a coloop, then

$$\chi_M(q) = (q-1)\, \chi_{M \setminus e}(q) = (q-1)\, \chi_{M/e}(q).$$

Otherwise,

$$\chi_M(q) = \chi_{M \setminus e}(q) - \chi_{M/e}(q).$$

Proof: If e is a coloop, then $M = M|e \oplus M\backslash e$
$\cong U_{1,1} \oplus M\backslash e$,

so
$$\chi_M(q) = \chi_{U_{1,1}}(q) \cdot \chi_{M\backslash e}(q)$$
$$= (q-1)\,\chi_{M\backslash e}(q).$$

$M\backslash e = M/e$
$\iff$ e is a loop or coloop

Assume now that is is not a coloop.
Then $rk(M\backslash e) = rk(M)$ and for any $S \subseteq E\backslash e$,

$$\begin{aligned} \cdot\, crk_{M\backslash e}(S) &= rk(M\backslash e) - rk_{M\backslash e}(S) \\ &= rk(M) - rk_M(S) \\ &= crk_M(S). \end{aligned}$$

Also, $rk(M/e) = rk(M) - 1$ and for any $S \subseteq E\backslash e$,

$$\begin{aligned} \cdot\, crk_{M/e}(S) &= rk(M/e) - rk_{M/e}(S) \\ &= (rk(M) - 1) - (rk_M(S \cup e) - \cancelto{1}{rk_M(e)}) \\ &= crk_M(S \cup e). \end{aligned}$$

So
$$\chi_M(q) = \sum_{S \subseteq E\backslash e} (-1)^{|S|} q^{crk_M(S)} + \sum_{S \subseteq E\backslash e} (-1)^{|S \cup e|} q^{crk_M(S \cup e)}$$
$$= \chi_{M\backslash e}(q) - \chi_{M/e}(q).$$

$\blacksquare$