Last time:

$$\chi_M(q) = 0$$ if M has a loop

$$\chi_M(q) = (q-1)\,\chi_{M\setminus e}(q).$$
$$= (q-1)\,\chi_{M/e}(q).$$
if e is a coloop

$$\chi_M(q) = \chi_{M\setminus e}(q) - \chi_{M/e}(q).$$
if e is not a loop or a coloop

Def: The chromatic polynomial of a graph G is the function

$$\mathrm{chr}_G : \mathbb{Z}_{\geq 0} \longrightarrow \mathbb{Z}_{\geq 0}$$
$$q \longmapsto \text{\# of proper } q\text{-colorings of } G$$

vertices colored with $\leq q$ colors s.t. adjacent vertices have different colors

Ex: If T is a tree on n vertices, then

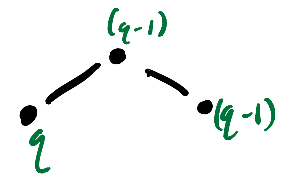
$$\mathrm{chr}_T(q) = q\,(q-1)^{n-1}$$

Proof: Remove a leaf + induct.

Ex: K_n complete graph on n vertices, then

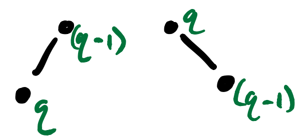
$$\mathrm{chr}_{K_n}(q) = q(q-1)(q-2)\cdots(q-n+1)$$

Ex: $\mathrm{chr}_G(q)$ does not depend only on $M(G)$.

$G_1 =$ 

$M(G_1) = U_{2,2}$

$\mathrm{chr}_{G_1}(q) = q(q-1)^2$

$G_2 =$ 

$M(G_2) = U_{2,2}$

$\mathrm{chr}_{G_2}(q) = q^2(q-1)^2$

Thm: $\mathrm{chr}_G(q) = q^c \chi_{M(G)}(q)$, where c is the number of connected components.

Proof: If G has a loop edge, then $\mathrm{chr}_G(q) = 0$ and $\chi_{M(G)}(q) = 0$.

If edges e and f are parallel in G, then $\mathrm{chr}_G(q) = \mathrm{chr}_{G \setminus e}(q)$, and $\chi_{M(G)}(q) = \chi_{M(G)\setminus e}(q)$.

So we may assume G is simple.

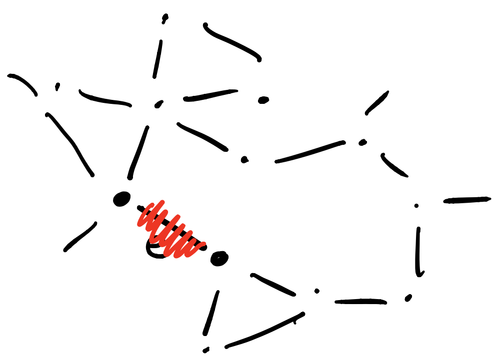
If $E(G) = \emptyset$ then $M(G) = U_{0,0}$ and

$$\mathrm{chr}_G(q) = q^{|V(G)|} = q^{|V(G)|} \underbrace{\chi_{U_{0,0}}(q)}_{=1}.$$

Otherwise, let $e \in E(G)$. Then

$$\mathrm{chr}_G(q) = \underbrace{\mathrm{chr}_{G\setminus e}(q)}_{\text{The coloring is proper except possibly at } e} - \underbrace{\mathrm{chr}_{G/e}(q)}_{\text{The coloring is proper except definitely at } e}.$$

Picture:



If e is not a coloop (isthmus / cut edge), then G, $G\setminus e$, and G/e all have the same number of conn. components. So by induction on $|E(G)|$,

$$\begin{aligned}\mathrm{chr}_G(q) &= q^c \chi_{M(G)\setminus e}(q) - q^c \chi_{M(G)/e}(q) \\ &= q^c \chi_{M(G)}(q)\end{aligned}$$

by the deletion-contraction formula for $\chi_{M(G)}$.

If e is a coloop, then $G \setminus e$ has one more component than G and G/e. So

$$\begin{aligned} chr_G(q) &= q^{c+1}\chi_{M(G)\setminus e}(q) - q^c \chi_{M(G)/e}(q) \\ &= q^c\left(q\,\chi_{M(G)\setminus e}(q) - \chi_{M(G)/e}(q)\right) \\ &= q^c(q-1)\chi_{M(G)\setminus e}(q) \\ &= q^c \chi_{M(G)}(q). \end{aligned}$$

$M \setminus e = M/e$ iff e is a loop or coloop

∎

Cor: F_7 is not graphic

Proof: $$\begin{aligned} q^c\chi_{F_7}(q) &= q^c(q^3 - 7q^2 + 14q - 8) \\ &= q^c(q-1)(q-2)(q-4) \end{aligned}$$

and this cannot be the chromatic polynomial of a graph.

$$chr_G(q) = q^c(q-1)(q-2)\cdots(q-k+1)\underbrace{}_{}$$ irreducible factors?

k = chromatic #.