Let M be a matroid, $r = rk(M)$. Then

$$\chi_M(q) = \sum_{S \subseteq E} (-1)^{|S|} q^{crk(S)}$$

$$= w_0 q^r + w_1 q^{r-1} + \cdots + w_{r-1} q + w_r .$$

The coefficients $w_i = w_i(M)$ are the Whitney numbers of the first kind.

Conjecture 1: For $0 \le i \le r$, $(-1)^i w_i > 0$.

Lecture 21, Exercise 2 - prove this using deletion-contraction formula.

Conjecture 2: The unsigned Whitney numbers of the first kind $|w_i| = (-1)^i w_i$ are unimodal:

$$|w_0| \le |w_1| \le \cdots \le |w_{k-1}| \le |w_k| \ge |w_{k+1}| \ge \cdots \ge |w_r| .$$

This would follow from the stronger property of log-concavity: $|w_i|^2 \ge |w_{i-1}| \cdot |w_{i+1}|$.

This was conjectured by Rota-Heron-Welsh in the 70s (Read conjectured it for chromatic polynomials), and proved by

Huh (2010) M rep'ble in char 0

Huh-Katz (2011) M rep'ble over some field

Adiprasito-Huh-Katz (2015) M arbitrary

Key idea: New perspective on χ_M inspired by geometry.

A condensed formula for χ_M

We have
$$\chi_M(q) = \sum_{S \subseteq E} (-1)^{|S|} q^{\mathrm{crk}(S)}$$

Since $\mathrm{rk}(S) = \mathrm{rk}(\mathrm{cl}(S))$ $\iff$ $\mathrm{crk}(S) = \mathrm{crk}(\mathrm{cl}(S))$

we have
$$\chi_M(q) = \sum_{F \in \mathcal{F}(M)} \underbrace{\left(\sum_{\substack{S \subseteq F \\ \mathrm{cl}(S) = F}} (-1)^{|S|} \right)}_{= U_F} q^{\mathrm{crk}(S)}$$

$$= \sum_{F \in \mathcal{F}(M)} U_F \, q^{\mathrm{crk}(F)}.$$

<u>Lemma</u>: If M has loops, then $U_F = 0$ for all $F \in \mathcal{F}(M)$.

Otherwise, $U_\emptyset = 1$ and

$$U_F = - \sum_{\substack{G \in \mathcal{F}(M) \\ G \subsetneq F}} U_G$$

for every nonempty flat F.

Proof: If e is a loop, $cl(S) = cl(S \cup e)$ for every $S \subseteq E \setminus e$. So for any flat F,

$$U_F = \sum_{\substack{S \subseteq F \\ cl(S)=F}} (-1)^{|S|} = \sum_{\substack{S \subseteq F\setminus e \\ cl(S)=F}} (-1)^{|S|} + \sum_{\substack{S \subseteq F\setminus e \\ cl(S)=F}} (-1)^{|S \cup e|}$$

$$= 0.$$

Otherwise, M is loopless and $U_\emptyset = (-1)^{|\emptyset|} = 1$.

If F is a nonempty flat, then

$$\sum_{\substack{G \in \mathcal{F}(M) \\ G \subseteq F}} U_G = \sum_{\substack{G \in \mathcal{F}(M) \\ G \subseteq F}} \sum_{\substack{S \subseteq G \\ cl(S) = G}} (-1)^{|S|}$$

$$= \sum_{S \subseteq F} (-1)^{|S|}$$

$S \subseteq F \iff cl(S) \subseteq F$

$$= 0$$

∎

More generally, we define the <u>Möbius function</u> of a poset P to be the unique function

$$\mu_P : P \times P \longrightarrow \mathbb{Z}$$

satisfying

- $\mu_P(x,x) = 1$ for all $x \in P$
- $\mu_P(x,y) = 0$ if $x \not\leq y$
- $\sum_{x \leq z \leq y} \mu_P(x,z) = 0$ if $x < y$

$$\iff \mu_P(x,y) = -\sum_{x \leq z \lneq y} \mu_P(x,z)$$

If we write $\mu_M = \mu_{\mathcal{L}(M)}$, then the lemma says

$$U_F = \sum_{\substack{S \subseteq F \\ cl(S) = F}} (-1)^{|S|} = \begin{cases} \mu_M(\emptyset, F) & \text{if } M \text{ loopless} \\ 0 & \text{otherwise.} \end{cases}$$

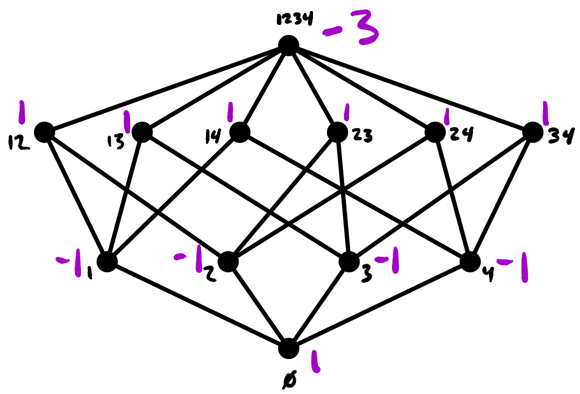
This is a version of Rota's <u>crosscut theorem</u>.

Cor: If M is loopless, then

$$\chi_M(q) = \sum_{F \in \mathcal{L}(M)} \mu_M(\emptyset, F)\, q^{\operatorname{crk}(F)}$$

Written as sum over $\mathcal{L}(M)$ b/c μ_M depends only on poset.

Ex: $M = U_{3,4}$



$\mu(\emptyset, F)$

$$\rightarrow \chi_{U_{3,4}}(q) = q^3 - 4q^2 + 6q - 3$$

Cor: The ith Whitney number of the first kind is

$$w_i(M) = w_i = \text{coeff. of } q^{r-i} \text{ in } \chi_M(q)$$

$$= \sum_{\substack{F \in \mathcal{L}(M) \\ \operatorname{rk}(F) = i}} \mu_M(\emptyset, F).$$